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A LINEARIZED DIFFERENCE SCHEME FOR A CLASS OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS WITH DELAY[1]

A class of non linear fractional partial differential equations with initial and Dirichlet boundary conditions is under consideration. We seek to obtain numerical solutions for this considered class of equations based on finite difference method. The convergence order will be $2-\alpha$ in time and four in space. A numerical example is given to support the theoretical results.

Keywords: fractional partial differential equation, linear difference scheme, delay, discrete energy method, convergence analysis.

Introduction

A great significance is devoted to study delay differential equations. They are widely used in many fields of science such as economics, physics, ecology, medicine, transportation scheduling, engineering control, computer aided design, nuclear engineering. They play a very important role in describing a variety of phenomena in the natural and social sciences. Also Fractional order differential equations, as generalizations of classical integer order differential equations, are increasingly used to model problems in fluid flow, finance and other areas of application. In [4, 5], numerical approximations for some different classes of fractional differential equations were discussed. There are many contributions in literature which deals with obtaining numerical solutions of space–time fractional partial differential equations such as [6]. This paper presents a practical linear difference scheme for solving space–time fractional partial differential equation with time delay. This linear difference scheme is applied previously for for a class of nonlinear delay partial differential equations [1, 7]. In this approach, we extend this idea to time and space fractional partial differential equation with nonlinear delay.

$$\frac{\partial^\alpha u}{\partial t^\alpha}-d\frac{\partial^\beta u}{\partial x^\beta}=f(x,t,u(x,t),u(x,t-s)),\quad a<x<b,\quad t\in[0,T], \tag{0.1}$$

$$u(a,t)=u_a(t),\quad u(b,t)=u_b(t),\quad t\in[0,T], \tag{0.2}$$

$$u(x,t)=\rho(x,t),\quad x\in[a,b],\quad t\in[-s,0), \tag{0.3}$$

where $0<\alpha<1$, $1<\beta\leqslant 2$, $d>0$ is the diffusion coefficient and $s>0$ is the delay parameter. Throughout this work, we suppose that the function $f(x,t,\mu,\nu)$ and the solution $u(x,t)$ are sufficiently smooth and assume that $f(x,t,\mu,\nu)$ has the first order continuous derivative with respect to the first and second components in the ϵ_0 neighborhood of the solution such that ϵ_0 is a positive constant. Let $c_0=\max\limits_{\substack{a<x<b\\ 0<t<T}}|\ u(x,t)\ |$, $c_1=\max\limits_{\substack{a<x<b,\,0<t<T\\ |\epsilon_1|\leqslant\epsilon_0,|\epsilon_2|\leqslant\epsilon_0}}|\ f_\mu(u(x,t)+\epsilon_1,\,u(x,t-s)+\epsilon_2,x,t)\ |$, $c_2=\max\limits_{\substack{a<x<b,\,0<t<T\\ |\epsilon_1|\leqslant\epsilon_0,|\epsilon_2|\leqslant\epsilon_0}}|\ f_\nu(u(x,t)+\epsilon_1,\,u(x,t-s)+\epsilon_2,x,t)\ |$.

§ 1. Derivation of the linearized difference scheme

Take two positive integers M and n, and let $h=\frac{b-a}{M}$, $\tau=\frac{s}{n}$ such that $x_i=a+ih$, $t_k=k\tau$ and $t_{k+\frac12}=(k+\frac12)\tau=\frac12(t_k+t_{k+1})$. Cover the domain by $\Omega_{h\tau}=\Omega_h\times\Omega_\tau$, where $\Omega_h=\{x_i|0\leqslant i\leqslant M\}$, $\Omega_\tau=\{t_k|-n\leqslant k\leqslant N\}$, $N=[\frac{T}{\tau}]$. Let $\mathcal{W}=\{\nu|\nu=\nu_i^k,0\leqslant i\leqslant M,-n\leqslant k\leqslant N\}$ be a grid function space on $\Omega_{h\tau}$. Define $\nu_i^{k+\frac12}=\frac12(\nu_i^k+\nu_i^{k+1})$.

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Kartary and his group [3] obtained the following approximation for the time Caputo fractional derivative at $t_{k+\frac{1}{2}}$:

$$\frac{\partial^\alpha u(t_{k+\frac{1}{2}}, x_i)}{\partial t^\alpha} = \Big[\omega_1 u^k + \sum_{m=1}^{k-1}(\omega_{k-m+1} - \omega_{k-m})u^m - \omega_k u^0 + \sigma\frac{(u_i^{k+1} - u_i^k)}{2^{1-\alpha}}\Big] + O(\tau^{2-\alpha}), \quad (1.1)$$

such that

$$\omega_i = \sigma((i + \frac{1}{2})^{1-\alpha} - (i - \frac{1}{2})^{1-\alpha}), \quad \sigma = \frac{1}{\Gamma(2-\alpha)}\frac{1}{\tau^\alpha}, \quad 0 < \alpha < 1. \quad (1.2)$$

Also, Sun and his group [2] presented the following averaging operator

$$\mathfrak{A}\nu_x = c_2^\beta \nu(x-h) + (1 - 2c_2^\beta)\nu(x) + c_2^\beta \nu(x+h), \quad 1 < \beta \leqslant 2. \quad (1.3)$$

It is easy to verify that

$$\mathfrak{A}\nu(x) = (1 + c_2^\beta h^2 \delta_x^2)\nu(x). \quad (1.4)$$

Also,

$$\mathfrak{A}\Big(\frac{\partial^\beta}{\partial x^\beta}\nu(x)\Big) = \delta_x^\beta \nu(x) + O(h^4), \quad (1.5)$$

$$\delta_x^2 \nu(x) = \frac{1}{h^2}\Big(\nu(x+h) - 2\nu(x) + \nu(x-h)\Big),$$

$$\delta_x^\beta \nu(x) = \frac{1}{h^\beta}\sum_{k=0}^{+\infty} w_k^\beta \nu(x - (k-1)h) + O(h^2),$$

where $w_0^\beta = \lambda_1 g_0^\beta$, $w_1^\beta = \lambda_1 g_1^\beta + \lambda_0 g_0^\beta$, $w_k^\beta = \lambda_1 g_k^\beta + \lambda_0 g_{k-1}^\beta + \lambda_{-1} g_{k-2}^\beta$, $k \geqslant 2$,

$$\lambda_1 = \frac{\beta^2 + 3\beta + 2}{12}, \quad \lambda_0 = \frac{4 - \beta^2}{6}, \quad \lambda_{-1} = \frac{\beta^2 - 3\beta + 2}{12}, \quad c_2^\beta = \frac{-\beta^2 + \beta + 2}{24},$$

$$g_0^\beta = 1, \quad g_k^\beta = (1 - \frac{\beta + 1}{k})g_{k-1}^\beta.$$

They proved some properties concerned with the averaging operator $\mathfrak{A}$

$$\langle \mathfrak{A}u, \nu \rangle = \langle u, \mathfrak{A}\nu \rangle, \quad |\nu\|_{\mathfrak{A}}^2 = \langle \mathfrak{A}\nu, \nu \rangle,$$

$$\frac{\|\nu\|^2}{3} \leqslant \|\nu\|_{\mathfrak{A}}^2 \leqslant \|\nu\|^2, \quad \langle \delta_x^\beta \nu, \nu \rangle \leqslant 0.$$

R e m a r k 1. Riemann–Liouville and Caputo operators have the following property

$${}_R D_t^\alpha u(x,t) = {}_c D_t^\alpha u(x,t) - \sum_{k=0}^{m-1}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}u^{(k)}(x,0), \quad m-1 < \alpha \leqslant m, \quad m = 1,2,3,\ldots. \quad (1.6)$$

According to (1.1) and the property (1.6), we can write Kartary approximation $0 < \alpha < 1$ at the points $t_{k+\frac{1}{2}}$ as follows

$${}_R D_t^\alpha u(x_i, t_{k+\frac{1}{2}}) = \Big[\omega_1 u^k + \sum_{m=1}^{k-1}(\omega_{k-m+1} - \omega_{k-m})u^m + \Big(\frac{t_{k+\frac{1}{2}}^{-\alpha}}{\Gamma(1-\alpha)} - \omega_k\Big)u^0 + \sigma\frac{(u_i^{k+1} - u_i^k)}{2^{1-\alpha}}\Big] + O(\tau^{2-\alpha}). \quad (1.7)$$

Consider Eq.(0.1) at the points $(x_i, t_{k+\frac{1}{2}})$, gives

$$\frac{\partial^\alpha u(x_i, t_{k+\frac{1}{2}})}{\partial t^\alpha} - d\frac{\partial^\beta u(x_i, t_{k+\frac{1}{2}})}{\partial x^\beta} = f(x_i, t_{k+\frac{1}{2}}, u(x_i, t_{k+\frac{1}{2}}), u(x_i, t_{k+\frac{1}{2}} - s)), \quad (1.8)$$

$$0 \leqslant i \leqslant M, \quad 0 \leqslant k \leqslant N-1.$$

R e m a r k 2. Taylor expansion yields

$$\frac{\partial^\beta u(x_i,t_{k+\frac{1}{2}})}{\partial x^\beta}=\frac{\partial^2}{\partial x^2}\Big(I^{2-\beta}u(x_i,t_{k+\frac{1}{2}})\Big), \tag{1.9}$$

$$\begin{aligned}\frac{\partial^\beta u(x_i,t_{k+\frac{1}{2}})}{\partial x^\beta} &= \frac{1}{2}\Big(\frac{\partial^2}{\partial x^2}I^{2-\beta}u(x_i,t_k)+\frac{\partial^2}{\partial x^2}I^{2-\beta}u(x_i,t_{k+1})\Big)+O(\tau^2), &(1.10)\\ &= \frac{1}{2}\Big(\frac{\partial^\beta u(x_i,t_k)}{\partial x^\beta}+\frac{\partial^\beta u(x_i,t_{k+1})}{\partial x^\beta}\Big)+O(\tau^2). &(1.11)\end{aligned}$$

R e m a r k 3. Taylor expansion yields

$$u(x_i,t_{k+\frac{1}{2}})=u_i^{k+\frac{1}{2}}\cong\frac{3}{2}u_i^k-\frac{1}{2}u_i^{k-1}+O(\tau^2), \tag{1.12}$$

$$u(x_i,t_{k+\frac{1}{2}}-s)=u_i^{k-n+\frac{1}{2}}\cong\frac{1}{2}u_i^{k+1-n}+\frac{1}{2}u_i^{k-n}+O(\tau^2). \tag{1.13}$$

After substitution with (1.7), (1.11), and (1.12), (1.13) into (1.8), we obtain

$$\begin{aligned}&\Big[\omega_1u_i^k+\sum_{m=1}^{k-1}(\omega_{k-m+1}-\omega_{k-m})u_i^m+\Big(\frac{t_{k+\frac{1}{2}}^{-\alpha}}{\Gamma(1-\alpha)}-\omega_k\Big)u_i^0+\sigma\frac{(u_i^{k+1}-u_i^k)}{2^{1-\alpha}}\Big]-\\ &-\frac{d}{2}\Big(\frac{\partial^\beta u(x_i,t_k)}{\partial x^\beta}+\frac{\partial^\beta u(x_i,t_{k+1})}{\partial x^\beta}\Big)=\\ &=f\Big(x_i,t_{k+\frac{1}{2}},\frac{3}{2}u_i^k-\frac{1}{2}u_i^{k-1},\frac{1}{2}u_i^{k+1-n}+\frac{1}{2}u_i^{k-n}\Big)+O(\tau^{2-\alpha})+O(\tau^2),\end{aligned} \tag{1.14}$$

such that

$$0\leqslant i\leqslant M,\quad 0\leqslant k\leqslant N-1.$$

By Operating with the averaging operator $\mathfrak{A}$ on both sides of (1.14), we have

$$\begin{aligned}&\mathfrak{A}\Big[\omega_1u_i^k+\sum_{m=1}^{k-1}(\omega_{k-m+1}-\omega_{k-m})u_i^m+\Big(\frac{t_{k+\frac{1}{2}}^{-\alpha}}{\Gamma(1-\alpha)}-\omega_k\Big)u_i^0+\sigma\frac{(u_i^{k+1}-u_i^k)}{2^{1-\alpha}}\Big]-\\ &-\frac{d}{2}\mathfrak{A}\Big(\frac{\partial^\beta u(x_i,t_k)}{\partial x^\beta}+\frac{\partial^\beta u(x_i,t_{k+1})}{\partial x^\beta}\Big)=\\ &=\mathfrak{A}f\Big(x_i,t_{k+\frac{1}{2}},\frac{3}{2}u_i^k-\frac{1}{2}u_i^{k-1},\frac{1}{2}u_i^{k+1-n}+\frac{1}{2}u_i^{k-n}\Big)+O(\tau^{2-\alpha})+O(\tau^2).\end{aligned} \tag{1.15}$$

Recall the properties of the averaging operator $\mathfrak{A}$ (1.3)–(1.5), then (1.15) can be written as follows

$$\begin{aligned}&\mathfrak{A}\Big[\omega_1u_i^k+\sum_{m=1}^{k-1}(\omega_{k-m+1}-\omega_{k-m})u_i^m+\Big(\frac{t_{k+\frac{1}{2}}^{-\alpha}}{\Gamma(1-\alpha)}-\omega_k\Big)u_i^0+\sigma\frac{(u_i^{k+1}-u_i^k)}{2^{1-\alpha}}\Big]=\\ &=d\delta_x^\beta u_i^{k+\frac{1}{2}}+\mathfrak{A}f\Big(x_i,t_{k+\frac{1}{2}},\frac{3}{2}u_i^k-\frac{1}{2}u_i^{k-1},\frac{1}{2}u_i^{k+1-n}+\frac{1}{2}u_i^{k-n}\Big)+O(\tau^{2-\alpha})+O(\tau^2)+O(h^4).\end{aligned} \tag{1.16}$$

Then, we can write

$$\begin{aligned}&\mathfrak{A}\Big[\omega_1U_i^k+\sum_{m=1}^{k-1}(\omega_{k-m+1}-\omega_{k-m})U_i^m+\Big(\frac{t_{k+\frac{1}{2}}^{-\alpha}}{\Gamma(1-\alpha)}-\omega_k\Big)U_i^0+\sigma\frac{(U_i^{k+1}-U_i^k)}{2^{1-\alpha}}\Big]=\\ &=d\delta_x^\beta U_i^{k+\frac{1}{2}}+\mathfrak{A}f\Big(x_i,t_{k+\frac{1}{2}},\frac{3}{2}U_i^k-\frac{1}{2}U_i^{k-1},\frac{1}{2}U_i^{k+1-n}+\frac{1}{2}U_i^{k-n}\Big)+R_i^k,\end{aligned} \tag{1.17}$$

such that

$$1\leqslant i\leqslant M-1,\quad 0\leqslant k\leqslant N-1,$$

and

$$| R_i^k | \leqslant c_3\Big(\tau^{2-\alpha}+\tau^2+h^4\Big).$$

Noting that the initial and boundary conditions after partition will be:

$$U_0^k = u_a(t_k), \quad U_M^k = u_b(t_k), \quad 1\leqslant k\leqslant N, \tag{1.18}$$

$$U_i^k = \rho(x_i,t_k), \quad 0\leqslant i\leqslant M, \quad -n\leqslant k\leqslant 0. \tag{1.19}$$

Omit the small term R_i^k in (1.18) and replace U_i^k with u_i^k, the constructed linear difference scheme will have the following form

$$\mathfrak{A}\Big[\omega_1 u_i^k + \sum_{m=1}^{k-1}(\omega_{k-m+1}-\omega_{k-m})u_i^m + \Big(\frac{t_{k+\frac{1}{2}}^{-\alpha}}{\Gamma(1-\alpha)}-\omega_k\Big)u_i^0 + \sigma\frac{(u_i^{k+1}-u_i^k)}{2^{1-\alpha}}\Big] =$$

$$= d\delta_x^{\beta}u_i^{k+\frac{1}{2}} + \mathfrak{A}f\Big(x_i, t_{k+\frac{1}{2}}, \frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1}, \frac{1}{2}u_i^{k+1-n}+\frac{1}{2}u_i^{k-n}\Big), \tag{1.20}$$

such that

$$1\leqslant i\leqslant M-1, \quad 0\leqslant k\leqslant N-1,$$

$$u_0^k = u_a(t_k), \quad u_M^k = u_b(t_k), \quad 1\leqslant k\leqslant N, \tag{1.21}$$

$$u_i^k = \rho(x_i,t_k), \quad 0\leqslant i\leqslant M, \quad -n\leqslant k\leqslant 0. \tag{1.22}$$

Remark 4. When $\beta=2$, (1.20) coincides with the the linear difference scheme for the time fractional partial differential equation with delay

$$\frac{\partial^\alpha u}{\partial t^\alpha} - d\frac{\partial^2 u}{\partial x^2} = f(x,t,u(x,t),u(x,t-s)), \quad a<x<b, \quad t\in[0,T], \tag{1.23}$$

$$u(a,t)=u_a(t), \quad u(b,t)=u_b(t), \quad t\in[0,T], \tag{1.24}$$

$$u(x,t)=\rho(x,t), \quad x\in[a,b], \quad t\in[-s,0), \tag{1.25}$$

where $0<\alpha<1$, d is the diffusion coefficient and $s>0$ is the delay parameter.

And the resulted difference scheme will have the form

$$\mathbb{A}\Big[\omega_1 u_i^k + \sum_{m=1}^{k-1}(\omega_{k-m+1}-\omega_{k-m})u_i^m + \Big(\frac{t_{k+\frac{1}{2}}^{-\alpha}}{\Gamma(1-\alpha)}-\omega_k\Big)u_i^0 + \sigma\frac{(u_i^{k+1}-u_i^k)}{2^{1-\alpha}}\Big] =$$

$$= d\delta_x^{2}u_i^{k+\frac{1}{2}} + \mathbb{A}f\Big(x_i, t_{k+\frac{1}{2}}, \frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1}, \frac{1}{2}u_i^{k+1-n}+\frac{1}{2}u_i^{k-n}\Big). \tag{1.26}$$

The averaging operator $\mathbb{A}$ will have the following form

$$\mathbb{A}\nu(x) = (1+c_2^2h^2\delta_x^2)\nu(x) = (1+\frac{1}{12}h^2\delta_x^2)\nu(x) = c_2^2\nu(x-h)+(1-2c_2^2)\nu(x)+c_2^2\nu(x+h) =$$

$$= \frac{1}{12}\Big(\nu(x-h)+10\nu(x)+\nu(x+h)\Big).$$

Remark 5. If we replace the averaging operator $\mathfrak{A}$ by the unit operator I, then we obtain the following $2-\alpha$ order in time and second order in space difference scheme

$$\Big[\omega_1 u_i^k + \sum_{m=1}^{k-1}(\omega_{k-m+1}-\omega_{k-m})u_i^m + \Big(\frac{t_{k+\frac{1}{2}}^{-\alpha}}{\Gamma(1-\alpha)}-\omega_k\Big)u_i^0 + \sigma\frac{(u_i^{k+1}-u_i^k)}{2^{1-\alpha}}\Big] =$$

$$= d\delta_x^{\beta}u_i^{k+\frac{1}{2}} + f\Big(x_i, t_{k+\frac{1}{2}}, \frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1}, \frac{1}{2}u_i^{k+1-n}+\frac{1}{2}u_i^{k-n}\Big), \tag{1.27}$$

such that

$$1\leqslant i\leqslant M-1, \quad 0\leqslant k\leqslant N-1,$$

$$u_0^k = u_a(t_k), \quad u_M^k = u_b(t_k), \quad 1\leqslant k\leqslant N, \tag{1.28}$$

$$u_i^k = \rho(x_i,t_k), \quad 0\leqslant i\leqslant M, \quad -n\leqslant k\leqslant 0. \tag{1.29}$$

§ 2. Convergence and stability of the proposed scheme

Denote $e_i^k = U_i^k - u_i^k, \quad 0 \leqslant i \leqslant M, \quad -n \leqslant k \leqslant N$ and subtract (1.20)–(1.22) from (1.17)–(1.19), we obtain the error difference scheme

$$\mathfrak{A}\Big[\omega_1 e_i^k + \sum_{m=1}^{k-1}(\omega_{k-m+1} - \omega_{k-m})e_i^m + \Big(\frac{t_{k+\frac{1}{2}}^{-\alpha}}{\Gamma(1-\alpha)} - \omega_k\Big)e_i^0 + \sigma\frac{(e_i^{k+1} - e_i^k)}{2^{1-\alpha}}\Big] =$$

$$= d\delta_x^\beta e_i^{k+\frac{1}{2}} + \mathfrak{A}f\Big[\Big(x_i, t_{k+\frac{1}{2}}, \frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1}, \frac{1}{2}U_i^{k+1-n} + \frac{1}{2}U_i^{k-n}\Big) -$$

$$- f\Big(x_i, t_{k+\frac{1}{2}}, \frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1}, \frac{1}{2}u_i^{k+1-n} + \frac{1}{2}u_i^{k-n}\Big)\Big] + R_i^k, \tag{2.1}$$

$$1 \leqslant i \leqslant M-1, \quad 0 \leqslant k \leqslant N-1,$$

$$e_0^k = 0, \quad e_M^k = 0, \quad 1 \leqslant k \leqslant N, \tag{2.2}$$

$$e_i^k = 0, \quad 0 \leqslant i \leqslant M, \quad -n \leqslant k \leqslant 0. \tag{2.3}$$

If the spatial domain $[a, b]$ is covered by $\Omega_h = \{x_i \mid 0 \leqslant i \leqslant M,\}$ and let

$$V_h = \{\nu \mid \nu = (\nu_0, \ldots, \nu_M), \quad \nu_0 = \nu_M = 0\}$$

be a grid function space on Ω_h.

For any $u, \nu \in V_h$, define the discrete inner products and corresponding norms as

$$\langle u, \nu\rangle = h\sum_{i=1}^{M-1} u_i\nu_i, \quad \langle\delta_x u, \delta_x\nu\rangle = h\sum_{i=1}^{M}(\delta_x u_{i-\frac{1}{2}})(\delta_x\nu_{i-\frac{1}{2}}),$$

and

$$\| u \| = \sqrt{\langle u, u\rangle}, \quad | u |_1 = \sqrt{\langle\delta_x u, \delta_x u\rangle}, \quad \| u \|_\infty = \max_{0\leqslant i\leqslant M} | u |.$$

The following inequalities are achieved

$$\| u \|_\infty \leqslant \frac{\sqrt{b-a}}{2} | u |_1, \quad \| u \| \leqslant \frac{b-a}{\sqrt{6}} | u |_1. \tag{2.4}$$

L e m m a 2.1. *For any $u \in V_h$, it holds that*

$$\Big\langle\mathfrak{A}\Big[\omega_1 u_i^k + \sum_{m=1}^{k-1}(\omega_{k-m+1} - \omega_{k-m})u_i^m + \Big(\frac{t_{k+\frac{1}{2}}^{-\alpha}}{\Gamma(1-\alpha)} - \omega_k\Big)u_i^0 + \sigma\frac{(u_i^{k+1} - u_i^k)}{2^{1-\alpha}}\Big], u_i^{k+\frac{1}{2}}\Big\rangle \geqslant$$

$$\geqslant \frac{\sigma}{2^{2-\alpha}}\Big(\| u_i^{k+1} \|_{\mathfrak{A}}^2 - \| u_i^k \|_{\mathfrak{A}}^2 \Big).$$

L e m m a 2.2 ([7]). *Suppose that $\{F^k \mid k \geqslant 0\}$ be a non negative consequence and satisfies $F^{k+1} \leqslant A + B\tau\sum_{l=1}^k F^L, \quad k = 0, 1, \ldots,$ then $F^{k+1} \leqslant A\exp(Bk\tau), \quad k = 0, 1, \ldots,$ such that A, B are non negative constants.*

For the difference scheme (1.20)–(1.22) and by using the previous lemmas, we can deduce the following convergence result.

T h e o r e m 2.1. *Let $u(x, t), \quad x \in [a, b], \quad -s \leqslant t \leqslant T$ be the solution of* (0.1)–(0.3) *and $\{u_i^k \mid 0 \leqslant i \leqslant M, \quad -n \leqslant k \leqslant N\}$ be the solution of the considered difference scheme* (1.20)–(1.22), *denote $e_i^k = U_i^k - u_i^k, \quad 0 \leqslant i \leqslant M, \quad -n \leqslant k \leqslant N$ and*

$$C = \frac{M\sqrt{b-a}}{\epsilon}c_3\exp(\frac{M^2(10\epsilon^2 + 5c_1^2 + c_2^2)}{6\epsilon^2}), \quad \epsilon = \frac{1}{3T\Gamma(2-\alpha)2^{2-\alpha}},$$

then if

$$\tau \leqslant \Big(\frac{\epsilon_0}{4C}\Big)^{\frac{1}{2-\alpha}}, \quad h \leqslant \Big(\frac{\epsilon_0}{4C}\Big)^{\frac{1}{4}},$$

we have

$$\| e^k \|_\infty \leqslant C\Big(\tau^{2-\alpha} + h^4\Big), \quad 0 \leqslant k \leqslant N. \tag{2.5}$$

To discuss the stability of the difference scheme (1.20)–(1.22), we use the discrete energy method in the same way like the discussion of the convergence. Let $\{\nu_i^k \mid 0 \leqslant i \leqslant M, \quad 0 \leqslant k \leqslant N\}$ be the solution of

$$\mathfrak{A}\Big[\omega_1\nu_i^k + \sum_{m=1}^{k-1}(\omega_{k-m+1}-\omega_{k-m})\nu_i^m + \Big(\frac{t_{k+\frac{1}{2}}^{-\alpha}}{\Gamma(1-\alpha)} - \omega_k\Big)\nu_i^0 + \sigma\frac{(\nu_i^{k+1}-\nu_i^k)}{2^{1-\alpha}}\Big] =$$

$$= d\delta_x^\beta\nu_i^k + \mathfrak{A}f\Big(x_i, t_{k+\frac{1}{2}}, \frac{3}{2}\nu_i^k - \frac{1}{2}\nu_i^{k-1}, \frac{1}{2}\nu_i^{k+1-n} + \frac{1}{2}\nu_i^{k-n}\Big), \tag{2.6}$$

such that

$$1 \leqslant i \leqslant M-1, \quad 0 \leqslant k \leqslant N-1,$$

$$\nu_0^k = u_a(t_k), \quad \nu_M^k = u_b(t_k), \quad 1 \leqslant k \leqslant N, \tag{2.7}$$

$$\nu_i^k = \rho(x_i, t_k) + \phi_i^k, \quad 0 \leqslant i \leqslant M, \quad -n \leqslant k \leqslant 0, \tag{2.8}$$

where ϕ_i^k is the perturbation of $\rho(x_i, t_k)$.

T h e o r e m 2.2. *Let* $\eta_i^k = \nu_i^k - u_i^k, \quad 0 \leqslant i \leqslant M, \quad -n \leqslant k \leqslant N$. *Then there exist constants* c_7, c_8, h_0, τ_0 *such that*

$$\| \eta^k \|_\infty \leqslant c_7\tau\sum_{k=-n}^{0} \| \phi^k \|, \quad 0 \leqslant k \leqslant N, \quad \| \phi^k \| = \sqrt{h\sum_{i=1}^{M-1}(\phi_i^k)^2},$$

only if

$$h \leqslant h_0, \quad \tau \leqslant \tau_0$$

and

$$\max_{\substack{-n\leqslant k\leqslant 0\\ 0\leqslant i\leqslant M}} | \phi_i^k | \leqslant c_8.$$

§ 3. Test example

Consider the following time-space fractional partial differential equation with delay

$$\frac{\partial^\alpha u}{\partial t^\alpha} - 2\frac{\partial^\beta u}{\partial x^\beta} = f(x, t, u(x,t), u(x, t-0.1)), \quad 1 < x < 2, \quad t \in (0,1], \tag{3.1}$$

$$u(1,t) = \frac{-31}{32}(t^3 - 2t - 1), \quad u(2,t) = 0, \quad t \in (0,1], \tag{3.2}$$

$$u(x,t) = (\frac{1}{32}x^6 - x)(t^3 - 2t - 1), \quad x \in (1,2), \quad t \in [-0.1, 0), \tag{3.3}$$

where $0 < \alpha < 1, \quad 1 < \beta \leqslant 2$,

$$f(x,t,u(x,t),u(x,t-0.1)) = u(x,t-0.1)^2 - 2\xi_1 + \xi_2 - (\frac{1}{32}x^6 - x)^2((t-0.1)^3 - 2(t-0.1) - 1)^2,$$

such that

$$\xi_1 = (\frac{1}{32}\frac{\Gamma(7)}{\Gamma(7)-\beta}x^{6-\beta} - \frac{\Gamma(2)}{\Gamma(2-\beta)}x^{1-\beta})(t^3 - 2t - 1),$$

$$\xi_2 = (\frac{\Gamma(4)}{\Gamma(4-\alpha)}t^{3-\alpha} - \frac{2\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha})(\frac{1}{32}x^6 - x).$$

The exact solution is

$$u(x,t) = (\frac{1}{32}x^6 - x)(t^3 - 2t - 1).$$

Let $u_i^k \mid 0 \leqslant i \leqslant M, 0 \leqslant k \leqslant N$ is the solution of the constructed difference scheme (1.20)–(1.22), define the maximum norm error

$$E_\infty(h,\tau) = \max_{\substack{0\leqslant i\leqslant M\\ 0\leqslant k\leqslant N}} | u(x_i, t_k) - u_i^k |.$$

In the following table, we present the maximum errors for different numerical solutions obtained with different step sizes when ($\alpha = 0.1$, $\beta = 1.9$).

h	τ	$E_\infty(h,\tau)$	$\log_2 \frac{E_\infty(h,\tau)}{E_\infty(h/2,\tau/4)}$
$\frac{1}{10}$	$\frac{1}{100}$	3.25×10^{-5}	3.96578
$\frac{1}{20}$	$\frac{1}{400}$	2.08×10^{-6}	3.98894
$\frac{1}{40}$	$\frac{1}{1600}$	1.310×10^{-7}	3.99516
$\frac{1}{80}$	$\frac{1}{6400}$	8.215×10^{-9}	*

§ 4. Conclusion

This work is related to a class of fractional partial differential equations with non linear delay. A linearized difference scheme was constructed to solve this sort of equations. Un conditional convergence and stability for the numerical difference scheme were proved. A numerical example supported our theoretical results. Our difference scheme can be easily applied for two dimensional delay problems with fractional orders.

REFERENCES

1. Ferreira J. Energy estimates for delay diffusion-reaction equations, *J. Comp. Math.*, 2008, vol. 26, no. 4, pp. 536–553.
2. Hao Z., Sun Z., Cao W. A fourth order approximation of fractional derivatives with its applications, *J. Comp. Phys.*, 2015, vol. 281, pp. 787–805. DOI: 10.1016/j.jcp.2014.10.053
3. Karatay I., Kale N., Bayramoglu S. A new difference scheme for time fractional heat equations based on Crank–Nicolson method, *Fract. Calc. Appl. Anal.*, 2013, vol. 16, no. 4, pp. 893–910. DOI: 10.2478/s13540-013-0055-2
4. Khader M.M., El Danaf T.S., Hendy S.A. A computational matrix method for solving systems of high order fractional differential equations, *Appl. Math. Model.*, 2013, vol. 37, pp. 4035–4050.
5. Pimenov G.V., Hendy S.A. Numerical studies for fractional functional differential equations with delay based on BDF-Type shifted Chebyshev polynomials, *Abstract and Applied Analysis*, 2015, vol. 2015, article ID 510875. DOI: 10.1155/2015/510875
6. Zhang Y. A finite difference method for fractional partial differential equation, *Appl. Math. Comp.*, 2009, vol. 16, no. 2, pp. 524–529.
7. Zhang Z., Sun Z. A linearized compact difference scheme for a class of nonlinear delay partial differential equations, *Appl. Math. Model.*, 2013, vol. 37, no. 3, pp. 742–752. DOI: 10.1016/j.apm.2012.02.036

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Линеаризованная разностная схема для класса дифференциальных уравнений с частными производными дробного порядка с запаздыванием

Ключевые слова: дифференциальные уравнения с частными производными дробного порядка, линейная разностная схема, запаздывание, анализ сходимости.

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