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NEW GENERALIZED INTEGRAL INEQUALITIES VIA (H, M)-CONVEX MODIFIED FUNCTIONS

In this article, we establish several inequalities for (h, m)-convex maps, related to weighted integrals, used in previous works. Throughout the work, we show that our results generalize several of the integral inequalities known from the literature.

Keywords: Hermite–Hadamard inequality, Hölder inequality, power mean inequality, weighted integrals, (m, h)-convex functions

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Introduction

One of the basic concepts of a number of mathematical disciplines (function theory, optimization theory, theory of inequalities, etc.) is the concept of convexity. This concept is closely related to the estimation of the mean value of a function given on an interval. The definition of convexity of a function and the central inequality used to estimate the mean value of a function are given in the literature as follows:

D e f i n i t i o n 0.1. The function $\phi\colon [\upsilon_1, \upsilon_2] \to \mathbb{R}$, is said to be convex, if we have

$$\phi\left(\varepsilon x + (1-\varepsilon) y\right) \leq \varepsilon \phi(x) + (1-\varepsilon)\phi(y)$$

$\forall\ x, y \in [\upsilon_1, \upsilon_2]$ and $\varepsilon \in [0, 1]$.

For convex functions on the interval $[\upsilon_1, \upsilon_2]$ at the end of the 19th century, the Hermite–Hadamard inequality

$$\phi\left(\frac{\upsilon_1+\upsilon_2}{2}\right) \leq \frac{1}{\upsilon_2-\upsilon_1}\int_{\upsilon_1}^{\upsilon_2} \phi(\varepsilon)\, d\varepsilon \leq \frac{\phi(\upsilon_1)+\phi(\upsilon_2)}{2}$$

widely known in the theory of inequalities, was established. Both inequalities hold in the reversed direction if ϕ is concave.

Today, in the literature there are quite a few different convexity classes of functions that are defined to refine, extend and improve estimates of the mean value of a function. A rather wide spectrum of convexity classes and their relations are given in [27].

Many researchers (see [2, 4–6, 11, 13, 14, 22, 25, 27, 28, 30, 33, 38–40] and references therein) have considered inequalities of the Hermite–Hadamard type with the use of integral operators for the purpose of generalization, refinement, and improvement.

For example, Bayraktar et al. in [4], Bayraktar and Özdemir in [5], by dividing the interval of integration into subintervals, obtained generalized inequalities for convex and s-convex functions were obtained in terms of fractional integration operators. Bessenyei and Páles in [6] studied generalized convex functions of high order, and, as a result, obtained extensions of the classical Hermite–Hadamard inequality. Using k-fractional integrals Butt et al. in [9] for the s-Godunov–Levin, quasi-convex and η-quasi-convex functions, new integral inequalities were obtained. Du et al. in [12] introduced the concept of generalized semi-(m, h)-preinvex functions and obtained some estimates for the trapezoid inequality. Nàpoles et al. in [25] obtained new integral inequalities of the Hermite–Hadamard type for convex and quasi-convex functions in a generalized context. In [26], Nápoles and Bayraktar established these inequalities for h-convex functions in the framework of a certain generalized integral. Xi and Qi in [39] and Yildiz et al. in [40] obtained new integral inequalities for s-convex, (α, s)-convex and extended s-convex functions.

We presented the following definitions.

D e f i n i t i o n 0.2. Let $h\colon [0,1] \to \mathbb{R}$ be a nonnegative function, $h \neq 0$ and $\phi\colon I = [0,+\infty) \to [0,+\infty)$. If inequality

$$\phi\left(\varepsilon\xi + m(1-\varepsilon)\varsigma\right) \leq h^s(\varepsilon)\phi(\xi) + m(1-h^s(\varepsilon))\phi(\varsigma)$$

is fulfilled for all $\xi, \varsigma \in I$ and $\varepsilon \in [0,1]$, where $m \in [0,1]$, $s \in (0,1]$, then it is said that function ϕ is (h,m)-convex modified of first type on I.

D e f i n i t i o n 0.3. Let $h\colon [0,1] \to \mathbb{R}$ nonnegative functions, $h \neq 0$ and $\phi\colon I = [0,+\infty) \to [0,+\infty)$. If inequality

$$\phi\left(\varepsilon\xi + m(1-\varepsilon)\varsigma\right) \leq h^s(\varepsilon)\phi(\xi) + m(1-h(\varepsilon))^s\phi(\varsigma)$$

is fulfilled for all $\xi, \varsigma \in I$ and $\varepsilon \in [0,1]$, where $m \in [0,1]$, $s \in [-1,1]$, then it is said that function ϕ is (h,m)-convex modified of second type on I.

R e m a r k 0.1. From Definitions 0.2 and 0.3 we can define $N^s_{h,m}[\upsilon_1, \upsilon_2]$, where $\upsilon_1, \upsilon_2 \in [0,+\infty)$, as the set of (h,m)-convex modified functions, for which $\phi(\upsilon_1) \geq 0$, characterized by the triple $(h(\varepsilon), m, s)$. Note that if:

1. $(h(\varepsilon), 0, 0)$, then we have increasing functions [8].
2. $(\varepsilon, 0, s)$, then we have s-starshaped functions [8].
3. $(\varepsilon, 0, 1)$, then we have starshaped functions [8].
4. $(\varepsilon, 1, 1)$, then ϕ is a convex function on $[0,+\infty)$ [8].
5. $(\varepsilon, m, 1)$, then ϕ is an m-convex function on $[0,+\infty)$ [36].
6. $(\varepsilon, 1, s)$ and $s \in (0,1]$, then ϕ is an s-convex function on $[0,+\infty)$ [7,15].
7. $(\varepsilon, 1, s)$ and $s \in [-1,1]$, then ϕ is an extended s-convex function on $[0,+\infty)$ [39].
8. (ε, m, s) and $s \in (0,1]$, then ϕ is an (s,m)-convex function on $[0,+\infty)$ [30].
9. $(\varepsilon^\alpha, m, 1)$ with $\alpha \in (0,1]$, then ϕ is an (α, m)-convex function on $[0,+\infty)$ [23].
10. $(\varepsilon^\alpha, m, s)$ with $\alpha \in (0,1]$, then ϕ is an s-(α, m)-convex function on $[0,+\infty)$ [38].
11. $(h(\varepsilon), m, 1)$, then we have a variant of an (h,m)-convex function on $[0,+\infty)$ [28].

R e m a r k 0.2. In the different notions of convexity, if the direction of the inequality changes, it will be called concave.

In our work, we use the Euler Gamma functions Γ (see [32]) and Γ_κ (see [10]):

$$\Gamma(z) = \int_0^\infty \varepsilon^{z-1} e^{-\varepsilon}\, d\varepsilon, \quad \mathrm{Re}\,(z) > 0,$$

$$\Gamma_\kappa(z) = \int_0^\infty \varepsilon^{z-1} e^{-\varepsilon^\kappa/\kappa}\, d\varepsilon, \quad \kappa > 0.$$

Here $\lim_{\kappa\to 1}\Gamma_\kappa(z)=\Gamma(z)$, $\Gamma_\kappa(z)=(\kappa)^{\frac{z}{\kappa}-1}\Gamma\left(\frac{z}{\kappa}\right)$ and $\Gamma_\kappa(z+\kappa)=z\Gamma_\kappa(z)$.

To encourage comprehension of the subject, we present the definition of Riemann–Liouville fractional integrals (with $0\le \upsilon_1<\varepsilon<\upsilon_2\le\infty$). The first one is the classical Riemann–Liouville fractional integral.

Definition 0.4. Let $\phi\in L_1[\upsilon_1,\upsilon_2]$. Then the Riemann–Liouville fractional integrals of order $\alpha\in\mathbb{C}$, $\mathrm{Re}\,(\alpha)>0$ are defined by (right and left respectively):

$$
\begin{aligned}
{}^{\alpha}I_{\upsilon_1^+}f(r)&=\frac{1}{\Gamma(\alpha)}\int_{\upsilon_1}^{r}(r-\varepsilon)^{\alpha-1}f(\varepsilon)\,d\varepsilon,\quad r>\upsilon_1,\\
{}^{\alpha}I_{\upsilon_2^-}f(r)&=\frac{1}{\Gamma(\alpha)}\int_{r}^{\upsilon_2}(\varepsilon-r)^{\alpha-1}f(\varepsilon)\,d\varepsilon,\quad r<\upsilon_2.
\end{aligned}
$$

Next we present the weighted integral operators, which will be the basis of our work.

Definition 0.5. Let $\phi\in L\left([\upsilon_1,\upsilon_2]\right)$ and let w be a continuous and positive function, $w\colon[0,1]\to[0,+\infty)$, with first order derivatives piecewise continuous on I. Then the weighted fractional integrals are defined by (right and left respectively):

$$
\begin{aligned}
J^{w}_{\upsilon_1^+}\phi(r)&=\int_{\upsilon_1}^{r}w'\left(\frac{\varepsilon-\upsilon_1}{r-\upsilon_1}\right)\phi(\varepsilon)\,d\varepsilon,\\
J^{w}_{\upsilon_2^-}\phi(r)&=\int_{r}^{\upsilon_2}w'\left(\frac{\upsilon_2-\varepsilon}{\upsilon_2-r}\right)\phi(\varepsilon)\,d\varepsilon,
\end{aligned}
$$

with $\upsilon_1<r\le\upsilon_2$.

Remark 0.3. To have a clearer idea of the amplitude of the Definition 0.5, let's consider some particular cases of the kernel w':

1. Putting $w'(\varepsilon)\equiv 1$, we obtain the classical Riemann integral.
2. If $w'(\varepsilon)=\frac{\varepsilon^{(\alpha-1)}}{\Gamma(\alpha)}$, then we obtain the right-sided Riemann–Liouville fractional integral, and the left-sided one can be obtained similarly.
3. With convenient kernel choices w' we can get the right and left k-Riemann–Liouville fractional integrals of [24], the right-sided fractional integrals of a function ϕ with respect to another function h on $[\upsilon_1,\upsilon_2]$ (see [2]), the right and left integral operator of [17], the right and left sided generalized fractional integral operators of [34] and the integral operators of [18] and [19], can also be obtained from above Definition by imposing similar conditions to w'.

Of course there are other known integral operators, fractional or not, that can be obtained as particular cases of the previous one, but we leave it to interested readers.

One of the most dynamic areas in Mathematical Sciences today is that of Integral Inequalities, mainly linked to the classical Hermite–Hadamard Inequality with the use of different integral, fractional or generalized operators. In particular, we can find different works with various particular operators (for example, see [1, 2, 9, 12, 16, 19, 21, 22, 26]). The advantage of using weighted integral operators is that it provides a unified and general formulation of various results that can be obtained as particular cases of ours. If we add to the above the definition of convexity used, we can realize the breadth and generality of the results obtained.

The main objective of this work is to obtain new variants of the classical Hermite–Hadamard Inequality, in the context of the weighted integrals of the Definition 0.5. We will show that our results complement or generalize several of those known from the literature.

§ 1. Results

To prove our theorems, we will need the following result.

L e m m a 1.1. *Let* $\phi\colon [\upsilon_1, \upsilon_2] \to \mathbb{R}$ *be a differentiable function,* $\upsilon_1 < r \leq \upsilon_2$, $\upsilon_2 > 0$. *If* $\phi \in L^1([\upsilon_1, \upsilon_2])$, *then we have:*

$$\begin{aligned} &\frac{(w(1)-w(0))(\phi(r)+\phi(\upsilon_1))}{2} - \frac{1}{2(r-\upsilon_1)}\left[J^w_{r-}\phi(\upsilon_1) + J^w_{\upsilon_1+}\phi(r)\right] \\ = &\ \frac{r-\upsilon_1}{2}\int_0^1 [w(1-\varepsilon)-w(\varepsilon)]\,\phi'(\varepsilon\upsilon_1+(1-\varepsilon)r)\,d\varepsilon. \end{aligned}$$

P r o o f. From known properties we have for the right member:

$$\begin{aligned} &\int_0^1 [w(1-\varepsilon)-w(\varepsilon)]\,\phi'(\varepsilon\upsilon_1+(1-\varepsilon)r)\,d\varepsilon \\ = &\int_0^1 w(1-\varepsilon)\phi'(\varepsilon\upsilon_1+(1-\varepsilon)r)\,d\varepsilon - \int_0^1 w(\varepsilon)\phi'(\varepsilon\upsilon_1+(1-\varepsilon)r)\,d\varepsilon \\ = &\ I_1 - I_2. \end{aligned}$$

For the first integral we have:

$$\begin{aligned} I_1 &= \int_0^1 w(1-\varepsilon)\phi'(\varepsilon\upsilon_1+(1-\varepsilon)r)\,d\varepsilon \\ &\left|\begin{aligned} u = w(1-\varepsilon) &\Longrightarrow du = -w'(1-\varepsilon)\,d\varepsilon \\ dv = \phi'(\varepsilon\upsilon_1+(1-\varepsilon)r)\,d\varepsilon &\Longrightarrow v = \frac{1}{\upsilon_1-r}\phi(\varepsilon\upsilon_1+(1-\varepsilon)r) \end{aligned}\right| \\ &= \left.\frac{w(1-\varepsilon)\phi(\varepsilon\upsilon_1+(1-\varepsilon)r)}{\upsilon_1-r}\right|_0^1 + \frac{1}{\upsilon_1-r}\int_0^1 w'(1-\varepsilon)\phi(\varepsilon\upsilon_1+(1-\varepsilon)r)\,d\varepsilon \\ &= \frac{w(0)\phi(\upsilon_1)-w(1)\phi(r)}{\upsilon_1-r} + \frac{1}{\upsilon_1-r}\int_0^1 w'(1-\varepsilon)\phi(\varepsilon\upsilon_1+(1-\varepsilon)r)\,d\varepsilon \end{aligned}$$

and changing the variable: $\varepsilon\upsilon_1+(1-\varepsilon)r = z$,

$$\begin{aligned} \varepsilon &= \frac{z-r}{\upsilon_1-r} \Longrightarrow d\varepsilon = \frac{dz}{\upsilon_1-r}, \\ 1-\varepsilon &= 1-\frac{z-r}{\upsilon_1-r} = \frac{\upsilon_1-z}{\upsilon_1-r} = \frac{z-\upsilon_1}{r-\upsilon_1}, \end{aligned}$$

we get

$$\begin{aligned} I_1 &= \frac{w(0)\phi(\upsilon_1)-w(1)\phi(r)}{\upsilon_1-r} + \frac{1}{\upsilon_1-r}\int_r^{\upsilon_1} w'\left(\frac{z-\upsilon_1}{r-\upsilon_1}\right)\phi(z)\,\frac{dz}{\upsilon_1-r} \\ &= -\frac{w(0)\phi(\upsilon_1)-w(1)\phi(r)}{r-\upsilon_1} - \frac{1}{(\upsilon_1-r)^2}\int_{\upsilon_1}^{r} w'\left(\frac{z-\upsilon_1}{r-\upsilon_1}\right)\phi(z)\,dz \\ &= \frac{w(1)\phi(r)-w(0)\phi(\upsilon_1)}{r-\upsilon_1} - \frac{1}{(r-\upsilon_1)^2}J^w_{\upsilon_1+}\phi(r). \end{aligned}$$

Similarly, for the I_2, we have

$$\begin{aligned} I_2 &= \int_0^1 w(\varepsilon)\phi'(\varepsilon\upsilon_1+(1-\varepsilon)r)\,d\varepsilon \\ &= -\frac{w(1)\phi(\upsilon_1)-w(0)\phi(r)}{r-\upsilon_1} + \frac{1}{(r-\upsilon_1)^2}J^w_{r-}\phi(\upsilon_1). \end{aligned}$$

Thus, we get

$$
\begin{aligned}
I_1 - I_2 &= -\frac{w(0)\phi(\upsilon_1) - w(1)\phi(r)}{r-\upsilon_1} - \frac{1}{(r-\upsilon_1)^2} J^w_{\upsilon_1+}\phi(r) \\
&\quad + \frac{w(1)\phi(\upsilon_1) - w(0)\phi(r)}{r-\upsilon_1} - \frac{1}{(r-\upsilon_1)^2} J^w_{r-}\phi(\upsilon_1) \\
&= \frac{[\phi(\upsilon_1)+\phi(r)]\,[w(1)-w(0)]}{r-\upsilon_1} - \frac{1}{(r-\upsilon_1)^2}\left[J^w_{\upsilon_1+}\phi(r) + J^w_{r-}\phi(\upsilon_1)\right].
\end{aligned}
$$

From $I_1 - I_2$ we obtain the desired result. This completes the proof. □

R e m a r k 1.1. Putting $w'(\varepsilon) = \frac{1}{\Gamma(\alpha)}\varepsilon^{\alpha-1}$, we obtain the Lemma 3 of [29], if $r = \upsilon_2$ the above result reduces to the Lemma 2 of [35]. With the kernel $w'(\varepsilon) = \frac{1}{k\Gamma_k(\alpha)}\varepsilon^{\frac{\alpha}{k}-1}$ if we put $r = \upsilon_2$ we get Lemma 2.3 of [14] (also see Lemma 2.1 of [37]).

R e m a r k 1.2. The main advantage of the previous result, over others known from the literature, in addition to the variety of "weights" $w'(\varepsilon)$ that we can consider (which can lead us to different integral operators), is the fact of the variability of the point $r \in (\upsilon_1, \upsilon_2)$, which obviously ensures a greater generality and strength of it.

R e m a r k 1.3. If we consider, $r = \frac{\upsilon_1+\upsilon_2}{2}$ we obtain the following results, not reported in the literature.

L e m m a 1.2. *Let $\phi\colon [\upsilon_1, \upsilon_2] \to \mathbb{R}$ be a differentiable function, $\upsilon_1 < r \leq \upsilon_2$, $\upsilon_2 > 0$. If $\phi \in L^1([\upsilon_1, \upsilon_2])$, then we have:*

$$
\begin{aligned}
&\frac{2}{\upsilon_2-\upsilon_1}\left\{\left[w(0)\phi(\frac{\upsilon_1+\upsilon_2}{2}) - w(1)\phi(\upsilon_1)\right] - \left[w(1)\phi(\upsilon_1) - w(0)\phi(\frac{\upsilon_1+\upsilon_2}{2})\right]\right\} \\
- &\frac{2}{(\upsilon_2-\upsilon_1)^2}\left(J^w_{\frac{\upsilon_1+\upsilon_2}{2}-}\phi(\upsilon_1) + J^w_{\upsilon_1+}\phi(\frac{\upsilon_1+\upsilon_2}{2})\right) \\
= &\int_0^1 [w(1-\varepsilon) - w(\varepsilon)]\,\phi'\left(\varepsilon\upsilon_1 + (1-\varepsilon)\frac{\upsilon_1+\upsilon_2}{2}\right) d\varepsilon.
\end{aligned}
$$

L e m m a 1.3. *Let $\phi\colon [\upsilon_1, \upsilon_2] \to \mathbb{R}$ be a differentiable function, $\upsilon_1 < r \leq \upsilon_2$, $\upsilon_2 > 0$. If $\phi \in L^1([\upsilon_1, \upsilon_2])$, then we have:*

$$
\frac{\phi(\upsilon_1) + \phi(\frac{\upsilon_1+\upsilon_2}{2})}{2} - \frac{1}{\upsilon_2-\upsilon_1}\int_{\upsilon_1}^{\frac{\upsilon_1+\upsilon_2}{2}} \phi(\varepsilon)\, d\varepsilon = \frac{\upsilon_2-\upsilon_1}{4}\int_0^1 (1-2\varepsilon)\phi(\varepsilon\upsilon_1 + \frac{\upsilon_1+\upsilon_2}{2}(1-\varepsilon))\, d\varepsilon.
$$

Considering $w(\varepsilon) = \varepsilon$, i. e., working with the classical Riemann integral.

T h e o r e m 1.1. *Let $0 \leq \upsilon_1 < \upsilon_2$, and $\phi \in C^1(\upsilon_1, \upsilon_2)$, where $\upsilon_1, \upsilon_2 \in \mathbb{R}$ such that $\upsilon_1 < r \leq \upsilon_2$, and ϕ be a function defined on the interval $[\upsilon_1, \upsilon_2]$. If $\phi' \in L[\upsilon_1, \upsilon_2]$ and $|\phi'|$ is a (h, m)-convex modified function of second type on $[\upsilon_1, \upsilon_2]$ for some fixed $m, s \in (0, 1]$, then the inequality*

$$
\begin{aligned}
&\left| A - \frac{1}{2(r-\upsilon_1)}\left(J^w_{\upsilon_1+}\phi(r) - J^w_{r-}\phi(\upsilon_1)\right)\right| \\
\leq\ &\frac{r-\upsilon_1}{2}\int_0^1 (w(1-\varepsilon) + w(\varepsilon))\left[|\phi'(\upsilon_1)|\, h^s(\varepsilon) + m\left|\phi'(\frac{r}{m})\right|(1-h(\varepsilon))^s\right] d\varepsilon,
\end{aligned}
$$

with $A = \frac{(w(1)-w(0))(\phi(r)+\phi(\upsilon_1))}{2}$, is fulfilled.

P r o o f. From Lemma 1.1 and taking into account the properties of modulus, we obtain

$$\begin{aligned} & \left|\upsilon_1 - \frac{1}{2(r-\upsilon_1)}\left(J^w_{\upsilon_1+}\phi(r) - J^w_{r-}\phi(\upsilon_1)\right)\right| \\ \leq & \frac{r-\upsilon_1}{2}\left|\int_0^1 \left[w(1-\varepsilon) - w(\varepsilon)\right]\phi'\left(\varepsilon\upsilon_1 + (1-\varepsilon)r\right)d\varepsilon\right| \qquad (1.1) \\ = & \frac{r-\upsilon_1}{2}\left\{\int_0^1 w(1-\varepsilon)\left|\phi'\left(\varepsilon\upsilon_1+(1-\varepsilon)r\right)\right|d\varepsilon + \int_0^1 w(\varepsilon)\left|\phi'\left(\varepsilon\upsilon_1+(1-\varepsilon)r\right)\right|d\varepsilon\right\} \end{aligned}$$

since $|\phi'|$ is an (h,m)-convex modified function of second type we have $|\phi'\left(\varepsilon\upsilon_1 + (1-\varepsilon)r\right)| \leq$ $\leq |\phi'(\upsilon_1)|\, h^s(\varepsilon) + m\left|\phi'(\frac{r}{m})\right|(1-h(\varepsilon))^s$, so

$$\begin{aligned} & \left\{\int_0^1 w(1-\varepsilon)\left|\phi'\left(\varepsilon\upsilon_1+(1-\varepsilon)r\right)\right|d\varepsilon + \int_0^1 w(\varepsilon)\left|\phi'\left(\varepsilon\upsilon_1+(1-\varepsilon)r\right)\right|d\varepsilon\right\} \\ \leq & |\phi'(\upsilon_1)|\int_0^1 w(1-\varepsilon)h^s(\varepsilon)\,d\varepsilon + m\left|\phi'(\frac{r}{m})\right|\int_0^1 w(1-\varepsilon)(1-h(\varepsilon))^s\,d\varepsilon \qquad (1.2) \\ + & |\phi'(\upsilon_1)|\int_0^1 w(\varepsilon)h^s(\varepsilon)\,d\varepsilon + m\left|\phi'(\frac{r}{m})\right|\int_0^1 w(\varepsilon)(1-h(\varepsilon))^s\,d\varepsilon. \end{aligned}$$

Substituting (1.2) into (1.1), rearranging and grouping, we get the desired inequality. □

R e m a r k 1.4. If we take $|\phi'|$ s-convex, i. e., $h(u) = u$, $m = 1$ and $w(\varepsilon) = \varepsilon^{\alpha}$, we get Theorem 5 of [40].

T h e o r e m 1.2. *Let $0 \leq \upsilon_1 < \upsilon_2$, and $\phi \in C^1(\upsilon_1, \upsilon_2)$, where $\upsilon_1, \upsilon_2 \in \mathbb{R}$ such that $\upsilon_1 <$ $< r \leq \upsilon_2$, and ϕ be a function defined on the interval $[\upsilon_1, \upsilon_2]$. If $\phi' \in L[\upsilon_1, \upsilon_2]$ and $|\phi'|^q$ is an (h,m)-convex modified function of second type on $[\upsilon_1, \upsilon_2]$ for some fixed $m, s \in (0,1]$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the inequality*

$$\begin{aligned} & \left|A - \frac{1}{2(r-\upsilon_1)}\left(J^w_{\upsilon_1+}\phi(r) - J^w_{r-}\phi(\upsilon_1)\right)\right| \\ \leq & \left(\frac{r-\upsilon_1}{2}\right)(B+C)\left[|\phi'(\upsilon_1)|^q\int_0^1 h^s(\varepsilon)d\varepsilon + m\left|\phi'(\frac{r}{m})\right|^q\int_0^1(1-h(\varepsilon))^s d\varepsilon\right]^{\frac{1}{q}} \end{aligned}$$

holds with A as before and $B = \left(\int_0^1 w^p(1-\varepsilon)\,d\varepsilon\right)^{\frac{1}{p}}$ and $C = \left(\int_0^1 w^p(\varepsilon)\,d\varepsilon\right)^{\frac{1}{p}}$.

P r o o f. As in the proof of the previous result, we have

$$\begin{aligned} & \left|A - \frac{1}{2(r-\upsilon_1)}\left(J^w_{\upsilon_1+}\phi(r) - J^w_{r-}\phi(\upsilon_1)\right)\right| \\ \leq & \frac{r-\upsilon_1}{2}\left|\int_0^1 \left[w(1-\varepsilon) - w(\varepsilon)\right]\phi'\left(\varepsilon\upsilon_1 + (1-\varepsilon)r\right)d\varepsilon\right| \qquad (1.3) \\ = & \frac{r-\upsilon_1}{2}\left\{\int_0^1 w(1-\varepsilon)\left|\phi'\left(\varepsilon\upsilon_1+(1-\varepsilon)r\right)\right|d\varepsilon + \int_0^1 w(\varepsilon)\left|\phi'\left(\varepsilon\upsilon_1+(1-\varepsilon)r\right)\right|d\varepsilon\right\} \end{aligned}$$

Using the Hölder inequality on the two integrals of (1.3), we get $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$:

$$\begin{aligned} & \int_0^1 w(1-\varepsilon)\left|\phi'\left(\varepsilon\upsilon_1+(1-\varepsilon)r\right)\right|d\varepsilon \\ \leq & \left(\int_0^1 w^p(1-\varepsilon)d\varepsilon\right)^{\frac{1}{p}}\left(\int_0^1 \left|\phi'\left(\varepsilon\upsilon_1+(1-\varepsilon)r\right)\right|^q d\varepsilon\right)^{\frac{1}{q}}, \qquad (1.4) \end{aligned}$$

$$\begin{aligned}&\int_0^1 w(\varepsilon)\left|\phi'\left(\varepsilon \upsilon_1+(1-\varepsilon)r\right)\right| d\varepsilon\\ \leq\ & \left(\int_0^1 w^p(\varepsilon)d\varepsilon\right)^{\frac{1}{p}}\left(\int_0^1 \left|\phi'\left(\varepsilon \upsilon_1+(1-\varepsilon)r\right)\right|^q d\varepsilon\right)^{\frac{1}{q}},\end{aligned} \tag{1.5}$$

using the (h,m)-convexity of $|\phi'|^q$ we obtain

$$\left(\int_0^1 \left|\phi'\left(\varepsilon \upsilon_1+(1-\varepsilon)r\right)\right|^q d\varepsilon\right)^{\frac{1}{q}} \leq \left(\left|\phi'(\upsilon_1)\right|^q h^s(\varepsilon)+m\left|\phi'(\frac{r}{m})\right|^q (1-h(\varepsilon))^s\right)^{\frac{1}{q}}, \tag{1.6}$$

so, using (1.6) in (1.4) and (1.5), and then in (1.3), rearranging, grouping and denoting $B = \left(\int_0^1 w^p(1-\varepsilon)\,d\varepsilon\right)^{\frac{1}{p}}$ and $C=\left(\int_0^1 w^p(\varepsilon)\,d\varepsilon\right)^{\frac{1}{p}}$, we arrive at the desired inequality. □

R e m a r k 1.5. As in the previous Remark, this result yields Theorem 6 of [40]. If, in addition, we put $r=\upsilon_2$, then we obtain Corollary 1 of the above-mentioned work.

R e m a r k 1.6. If we work with convex functions, i. e., $h(\varepsilon)=\varepsilon$, $s=m=1$, $r=\upsilon_2$ and $w(\varepsilon)=\varepsilon$, then the above result becomes Theorem 2.3 from [11]. Theorem 1 of [20] is also a particular case of this result.

T h e o r e m 1.3. *Let* $0\leq \upsilon_1<\upsilon_2$, *and* $\phi\in C^1(\upsilon_1,\upsilon_2)$, *where* $\upsilon_1,\upsilon_2\in\mathbb{R}$ *such that* $\upsilon_1 < r\leq \upsilon_2$ *and* ϕ *be a function defined on the interval* $[\upsilon_1,\upsilon_2]$. *If* $\phi'\in L[\upsilon_1,\upsilon_2]$ *and* $|\phi'|^q$ *is an* (h,m)*-convex modified function of second type on* $[\upsilon_1,\upsilon_2]$ *for some fixed* $m,s\in(0,1]$ *and* $q\geq 1$, *then the inequality*

$$\begin{aligned}&\left|A-\frac{1}{2(r-\upsilon_1)}\left(J^w_{\upsilon_1+}\phi(r)-J^w_{r-}\phi(\upsilon_1)\right)\right|\\ \leq\ & \left(\frac{r-\upsilon_1}{2}\right)\cdot D\cdot\left(\int_0^1 w(1-\varepsilon)\left(\left|\phi'(\upsilon_1)\right|^q h^s(\varepsilon)+m\left|\phi'(\frac{r}{m})\right|^q(1-h(\varepsilon))^s\right)d\varepsilon\right)^{\frac{1}{q}}\\ +\ & \left(\frac{r-\upsilon_1}{2}\right)\cdot E\cdot\left(\int_0^1 w(\varepsilon)\left(\left|\phi'(\upsilon_1)\right|^q h^s(\varepsilon)+m\left|\phi'(\frac{r}{m})\right|^q(1-h(\varepsilon))^s\right)d\varepsilon\right)^{\frac{1}{q}},\end{aligned}$$

holds, with A *as before and* $D=\left(\int_0^1 w(1-\varepsilon)\,d\varepsilon\right)^{1-\frac{1}{p}}$ *and* $E=\left(\int_0^1 w(\varepsilon)\,d\varepsilon\right)^{1-\frac{1}{p}}$.

P r o o f. As before

$$\begin{aligned}&\left|\upsilon_1-\frac{1}{2(r-\upsilon_1)}\left(J^w_{\upsilon_1+}\phi(r)-J^w_{r-}\phi(\upsilon_1)\right)\right|\\ \leq\ & \frac{r-\upsilon_1}{2}\left|\int_0^1 \left[w(1-\varepsilon)-w(\varepsilon)\right]\phi'\left(\varepsilon\upsilon_1+(1-\varepsilon)r\right)d\varepsilon\right|\\ =\ & \frac{r-\upsilon_1}{2}\left\{\int_0^1 w(1-\varepsilon)\left|\phi'\left(\varepsilon\upsilon_1+(1-\varepsilon)r\right)\right|d\varepsilon+\int_0^1 w(\varepsilon)\left|\phi'\left(\varepsilon\upsilon_1+(1-\varepsilon)r\right)\right|d\varepsilon\right\}\end{aligned}$$

Using now the well-known power mean inequality with modulus properties,we obtain:

$$\begin{aligned}&\int_0^1 w(1-\varepsilon)\left|\phi'\left(\varepsilon\upsilon_1+(1-\varepsilon)r\right)\right|d\varepsilon+\int_0^1 w(\varepsilon)\left|\phi'\left(\varepsilon\upsilon_1+(1-\varepsilon)r\right)\right|d\varepsilon\\ \leq\ & \left(\int_0^1 w(1-\varepsilon)d\varepsilon\right)^{1-\frac{1}{q}}\left(\int_0^1 w(1-\varepsilon)\left|\phi'(\varepsilon\upsilon_1+(1-\varepsilon)r)\right|d\varepsilon\right)^{\frac{1}{q}}\\ +\ & \left(\int_0^1 w(\varepsilon)d\varepsilon\right)^{1-\frac{1}{q}}\left(\int_0^1 w(\varepsilon)\left|\phi'(\varepsilon\upsilon_1+(1-\varepsilon)r)\right|d\varepsilon\right)^{\frac{1}{q}}.\end{aligned}$$

From the (h,m)-convexity of $|\phi'|^q$, and a simple but tedious algebraic work, the proof of the Theorem is completed. □

Remark 1.7. Assuming $|\phi'|^q$ s-convex and $w(\varepsilon)=\varepsilon^{\alpha}$, we get Theorem 7 of [40].

§ 2. Conclusions

In this paper, various extensions and generalizations of the classical Hermite–Hadamard Inequality have been established, in the context of recently defined weighted integral operators. Throughout it, various particular cases of the results obtained have been shown, which proves their breadth and strength. However, we do not want to conclude without pointing out two more aspects regarding the breadth of our results. Firstly, referring to the integral operator used, given that the weight function can include several known cases, we can add that if $w'(\varepsilon)=\varepsilon^{1-\frac{\alpha}{k}}$ (that is, we consider the k-integral of [15]), and $r=\upsilon_2$ the Lemma 1.1 reduces to Lemma 2.1 of [37], obviously many of the results of that work, can also be obtained from ours, considering convex functions. The second issue is the fact that the weighted operators used in our work can be used to derive new generalizations of other inequalities, for example, the Minkowski inequality (see, for example, [3]) where the weight $w'(\varepsilon)=\frac{1}{\Gamma(\alpha)}\varepsilon^{\alpha-1}$ is used. All of the above opens new and promising lines of work in this fruitful area.

The effectiveness of the method lies, fundamentally, in that we can provide a general working method that encompasses functions with first derivative, second or higher order derivatives, either with an integral operator, with two operators, on the same interval, on subintervals, without forgetting that it encompasses many of the known weights, used in various particular cases.

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Новые обобщенные интегральные неравенства через (h, m)-выпуклые модифицированные функции

Ключевые слова: неравенство Эрмита–Адамара, неравенство Гёльдера, степенное неравенство, взвешенные интегралы, (m, h)-выпуклые функции

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В этой статье мы устанавливаем несколько неравенств для (h, m)-выпуклых отображений, связанных с взвешенными интегралами, которые использовались в предыдущих работах. На протяжении всей работы мы показываем, что наши результаты обобщают несколько известных из литературы интегральных неравенств.

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