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DIFFERENTIAL GAME WITH "LIFELINE" FOR PONTRYAGIN'S CONTROL EXAMPLE

The main purpose of this work is to solve one of the main problems of Isaacs, i. e., a game with a "lifeline" for Pontryagin's control example when both players have the same movement dynamics. To solve this problem, the pursuer is offered a strategy of parallel pursuit (briefly, Π-strategy), which ensures the fastest convergence of the players and the capture of the evader within a certain closed ball. In addition, for the differential game under consideration, an explicit analytical formula for the players' attainability domain is given and the main lemma is generalized (L. A. Petrosjan's lemma on monotonicity of the players' attainability domain with respect to embedding for a game of simple pursuit). Using this main lemma, we find conditions for the solvability of the game with a "lifeline" for Pontryagin's control example as well. For clarity, at the end of the work, examples are given for some special cases.

Keywords: differential game, pursuit, evasion, acceleration, strategy, guaranteed capture time, attainability domain, lifeline.

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Introduction

The fundamental theory of differential games was formulated by R. Isaacs [1], L. S. Pontryagin [2], N. N. Krasovskii [3], B. N. Pshenichnyi [4], L. A. Petrosjan [5], A. Friedman [6], W. H. Fleming [7], L. D. Bercovitz [8], A. A. Chikrii [9], A. I. Subbotin [10] and others.

In the theory of differential games, pursuit–evasion problems occupy a special place due to a number of specific qualities. One of them is expansiveness of applications of various methods and originality of the obtained results [1,2,4,5,11,12]. This quality was clearly apparent in the model problems. For example, R. Isaacs' example called "a game with a lifeline" [1, Problem 9.5.1] with simple dynamics of the players was solved by L. A. Petrosjan introducing a special strategy [5] called strategy of parallel approach (briefly, Π-strategy). Later, Π-strategy was efficiently applied to solve other kinds of pursuit games [9, 13–18]. Later on, A. A. Chikrii [9] worked out the resolving functions on the basis of combination of ideas of Π-strategy and first direct method of L. S. Pontryagin [2].

A. A. Azamov based on B. N. Pshenichnyi's work [4] gave analytical formula for Π-strategy suited for all cases of maximal velocities of the players and inducted a recurrent relation for attainability domain [13]. Such a relation turned out effective in solving of pursuit–evasion problems and "a game with a lifeline" with different variants of constraints for control functions of the players [14, 15].

Another important differential game famous under the name "control example" was considered by L. S. Pontryagin and for this game, further important results were achieved in the works [11, 12, 16]. Differ from "a game with a lifeline", the Pontryagin control example deals with inertial objects and is therefore appreciably complicated. Equations of a pursuer and an evader in the example describe the motion of two inertial objects with allowance for friction forces. In [19], for this example, l-capture problem was solved by exchanging given differential equations to a linear normal system. In [2], Pontryagin's example with many players having the same dynamic and inertial resources was studied and proved that matching of the phase coordinates is a pursuit condition. In the work [12] of N. N. Petrov, a "soft" pursuit for the Pontryagin example of n pursuers and one evader was considered and proved that matching of the phase coordinates and that of the velocities are pursuit conditions. In [20], D. A. Vagin and N. N. Petrov obtained sufficient conditions for the capture of at least one evader for the Pontryagin example with many participants and phase constraints imposed on the position of evaders with identical dynamic and inertial resources of players, and with all evaders using the same control.

The problems in the theory of differential games are basically studied in the cases where control functions of players are subject to geometric, integral or their mixed constraints (for instance, [21–25]). Nevertheless, various type constraints on controls have been provoking a considerable interest in a number of applied problems such as economical, biological and ecological problems.

At the present time, from the standpoint of applied and theoretical perspective there is a substantial enthusiasm to explore various type problems of optimal control for more complex systems when different types of stationary and non-stationary constraints are imposed on the controls (for example, [26–28]). Differential games under phase constraints might be involved in such problems where the attainability domain of players has a considerable significance. Construction of the attainability domain is regarded as the remarkable result in the problems of avoidance of encounter [5] and in the conflict problems [29–34] as well. In the work [35], the authors investigated the differential game with a "lifeline" for the inertial movements under geometric constraints on controls of players and the dynamics of the attainability domain of an evader was examined through finding solvability conditions of the pursuit–evasion problems in favor of a pursuer or an evader.

In this paper, pursuit–evasion problems and a "lifeline" game are considered for Pontryagin's control example when both players have the same movement dynamics. In the pursuit problem, an analogue of Π-strategy is constructed and a sufficient condition of capture is determined. In the evasion problem, a constant control function is proposed as a strategy of an evader and a sufficient condition of evasion is found. Further, an explicit formula for the attainability domain of players is generated. Monotonicity conditions of the attainability domain with respect to embedding and a sufficient condition of defining a solution of the game with a "lifeline" are obtained.

§ 1. Formulation of the problems

We will consider a differential game of two players. Let in the Euclidean space $\mathbb{R}^n$, a controllable player X (the pursuer) follow another controllable player Y (the evader). Assume that positions of the players X and Y are described by x and y, respectively in $\mathbb{R}^n$.

Let movements of the pursuer and the evader be based on the following differential equations with initial conditions:

$$X\colon \ddot{x} + a\dot{x} = u, \quad x(0) = x_{10}, \quad \dot{x}(0) = x_{20}, \tag{1.1}$$

$$Y\colon \ddot{y} + a\dot{y} = v, \quad y(0) = y_{10}, \quad \dot{y}(0) = y_{20}, \tag{1.2}$$

where $x, y, u, v \in \mathbb{R}^n$, $n \geq 2$, $a > 0$; x_{10}, y_{10} are the initial positions of the players X and Y, and x_{20}, y_{20} are their initial velocity vectors. Here it is required that $x_{10} \neq y_{10}$ and $x_{20} = y_{20}$.

The parameter u in (1.1) is a control of the pursuer and it will be thought of as a measurable function $u(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}^n$. On this vector-function, we put the geometric constraint (briefly, the G-constraint) of the form

$$|u(t)| \leq \alpha \ \text{ almost everywhere } \ t \geq 0, \tag{1.3}$$

where α is a given positive parametric number and it expresses the maximal value of acceleration of the pursuer.

Likewise, the parameter v in (1.2) is a control of the evader and it will be regarded as a measurable function $v(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}^n$. On this vector-function, we put the G-constraint

$$|v(t)| \le \beta \ \text{ almost everywhere } \ t \ge 0, \tag{1.4}$$

where β is a given non-negative parametric number and it expresses the maximal value of acceleration of the evader.

Control functions $u(\cdot)$ and $v(\cdot)$ of the players depend on time $t,\ t \ge 0$. The set of all control functions $u(\cdot)$ ($v(\cdot)$) satisfying the constraint (1.3) (the constraint (1.4)) is denoted by $\mathbb{U}$ (by $\mathbb{V}$).

D e f i n i t i o n 1.1. A measurable function $u(\cdot) \in \mathbb{U}$ ($v(\cdot) \in \mathbb{V}$) is called *an admissible control of the pursuer (of the evader)*.

By equations (1.1) and (1.2), for controls $u(\cdot) \in \mathbb{U}$ and $v(\cdot) \in \mathbb{V}$, the triplets $(x_{10}, x_{20}, u(\cdot))$ and $(y_{10}, y_{20}, v(\cdot))$ pose the trajectories

$$x(t) = x_{10} + \frac{x_{20}}{a}\left(1 - e^{-at}\right) + \frac{1}{a}\int_0^t u(s)\left(1 - e^{-a(t-s)}\right)ds, \tag{1.5}$$

$$y(t) = y_{10} + \frac{y_{20}}{a}\left(1 - e^{-at}\right) + \frac{1}{a}\int_0^t v(s)\left(1 - e^{-a(t-s)}\right)ds \tag{1.6}$$

of the players X and Y, respectively.

Assume that a closed subset L called the "lifeline" is given in $\mathbb{R}^n$. The main target for the pursuer is to catch the evader, as it were, to obtain the equality $x(t_*) = y(t_*)$ at some $t_* > 0$ while the evader stays in the zone $\mathbb{R}^n \setminus L$. The main goal for the evader is to reach the zone L before being caught by the pursuer, or to sustain the relation $x(t) \ne y(t)$ at each $t \in [0, +\infty)$, and if this is not possible, then to extend an encounter time asap. We need to remark that the zone L does not restrict motion of the pursuer. In addition, it is supposed that the initial positions x_{10}, y_{10} satisfy the conditions $x_{10} \ne y_{10}$ and $y_{10} \notin L$ at the beginning of the game.

It is known that if control functions of the players X and Y depend only on time $t,\ t \ge 0$, then they don't guarantee to solve the games of pursuit and evasion. Thus, the acceptable types of controls involve being strategies. Below we are going to present the key definitions and conceptions.

First off, let us write the notations

$$z(t) = x(t) - y(t), \quad z(0) = z_{10} = x_{10} - y_{10}, \quad \dot z(0) = z_{20} = x_{20} - y_{20}. \tag{1.7}$$

Then according to (1.1), (1.2), (1.7), we get the initial value problem

$$\ddot z + a\dot z = u - v, \quad z(0) = z_{10}, \quad \dot z(0) = z_{20} = 0. \tag{1.8}$$

Consequently, as an alternative to the game (1.1)–(1.4), we have generated the game (1.3), (1.4), (1.8), or for brevity, the game ($\mathbb{U}$, $\mathbb{V}$).

Later, we will use the symbol B_ϱ to denote the ball of a radius ϱ centered at the origin.

D e f i n i t i o n 1.2. It is said that a function $\boldsymbol{u}\colon \mathbb{V} \times \mathbb{R}^n \to \mathbb{U}$ is *a strategy of the pursuer* if:

(1) for any control $v(\cdot) \in \mathbb{V}$, the inclusion $u(\cdot) = \boldsymbol{u}(v(\cdot), z_{10}) \in \mathbb{U}$ is satisfied. Here we term the function $u(\cdot) = \boldsymbol{u}(v(\cdot), z_{10})$ as *an implementation of the strategy* $\boldsymbol{u}(\cdot)$;

(2) for every $v_1(\cdot), v_2(\cdot) \in \mathbb{V}$ and for each $t,\ t \ge 0$, the equality $v_1(\varepsilon) = v_2(\varepsilon)$ is fulfilled almost everywhere on $[0, t]$, then $u_1(\varepsilon) = u_2(\varepsilon)$ is true almost everywhere on the same interval, where $u_i(\cdot) = \boldsymbol{u}(v_i(\cdot), z_{10})$, $i = 1, 2$.

A special case of a strategy can be given by a mapping $\boldsymbol{u}\colon B_\beta \times \mathbb{R}^n \to B_\alpha$, which is a Borel measurable function with respect to v, $v \in B_\beta$.

D e f i n i t i o n 1.3. We say that a strategy $\boldsymbol{u} = \boldsymbol{u}(v, z_{10})$ is *a parallel pursuit strategy*, or briefly, *a* Π*-strategy* if, for any control $v(\cdot) \in \mathbb{V}$, the solution $z(t)$ of Cauchy's problem

$$\ddot{z} + a\dot{z} = \boldsymbol{u}(v(t), z_{10}) - v(t), \quad z(0) = z_{10}, \quad \dot{z}(0) = 0$$

can be transformed into the form

$$z(t) = \Gamma(v(\cdot), t)z_{10}, \quad \Gamma(v(\cdot), 0) = 1,$$

where $\Gamma(v(\cdot), t)$ is a scalar function with respect to $t,\ t \geq 0$, and in general, this function is called *an approach function* in the game of pursuit.

D e f i n i t i o n 1.4. A Π-strategy is said to be *winning for the pursuer* on the interval $[0, T(\boldsymbol{u})]$ in the game $(\mathbb{U}, \mathbb{V})$ beginning from (z_{10}, z_{20}) if, for any control $v(\cdot) \in \mathbb{V}$, there exists some time $t^* \leq T(\boldsymbol{u})$ such that $z(t^*) = 0$. If this property is valid, then $T(\boldsymbol{u})$ is called *a guaranteed capture time*.

D e f i n i t i o n 1.5. A control function $\boldsymbol{v}_*(t) : \mathbb{R}_+ \to \mathbb{V}$ is called *a strategy of the evader* if $\boldsymbol{v}_*(t)$ is a Lebesgue measurable function with t, $t \geq 0$.

Now we are going to treat the game $(\mathbb{U}, \mathbb{V})$ from the perspective of the player Y.

D e f i n i t i o n 1.6. A strategy $\boldsymbol{v}_*(\cdot) \in \mathbb{V}$ is said to be *winning for the evader* in the game $(\mathbb{U}, \mathbb{V})$ beginning from (z_{10}, z_{20}) if, for any control $u(\cdot) \in \mathbb{U}$, the solution $z(t)$ of Cauchy's problem

$$\ddot{z} + a\dot{z} = u(t) - \boldsymbol{v}_*(t), \quad z(0) = z_{10}, \quad \dot{z}(0) = 0,$$

is not zero, i.e., $z(t) \neq 0$ at each $t \in [0, +\infty)$.

In this article, the following game problems will be individually investigated:

P r o b l e m 1.1. The problem of pursuit. To construct a Π-strategy for the pursuer and to find a sufficient condition of capture.

P r o b l e m 1.2. The problem of evasion. To set an optimal strategy for the evader and to estimate how to alter the distance $|z(t)|$ between the pursuer and the evader.

P r o b l e m 1.3. To construct an attainability domain of the pursuer (the set of capture points).

P r o b l e m 1.4. To solve the "lifeline" game.

§ 2. Solution of the problem of pursuit

In the present section, the Π-strategy will be defined on the basis of the works [4,5,13–15,35] and a sufficient condition of capture will be taken.

If the players X and Y single out their admissible control functions $u(\cdot) \in \mathbb{U}$ and $v(\cdot) \in \mathbb{V}$, respectively, then using (1.5)–(1.7), obtain the vector-function

$$z(t) = z_{10} + \frac{1}{a}\int_0^t \big(u(s) - v(s)\big)\Big(1 - e^{-a(t-s)}\Big)\, ds. \tag{2.1}$$

On account of (2.1), the pursuer aims for the attainment of the equality $z(t_*) = 0$ at some $t_* > 0$, while the evader strives to continue the relation $z(t) \neq 0$ for each t, $t \geq 0$.

Consider the function

$$p(t) = e^{-at} + at - b, \quad b = 1 + \frac{a^2|z_{10}|}{\alpha - \beta}. \tag{2.2}$$

The following assertion will be used to show the existence of capture time.

P r o p o s i t i o n 2.1. *Suppose $\alpha > \beta$. Then the equation*

$$p(t) = 0, \quad t \geq 0, \tag{2.3}$$

has one and only one positive root that will be denoted by $\widehat{T}$.

To construct the Π-strategy, suppose that the pursuer is aware of the initial data z_{10}, α, β and the value $v(t)$ at the current time t.

D e f i n i t i o n 2.1. We say that the vector-function

$$\boldsymbol{u}_\Pi(v, z_{10}) = v - \gamma(v, z_{10})\xi_{10}, \tag{2.4}$$

is *the* Π-*strategy* in the game $(\mathbb{U}, \mathbb{V})$, where

$$\gamma(v, z_{10}) = \langle v, \xi_{10}\rangle + \sqrt{\langle v, \xi_{10}\rangle^2 + \alpha^2 - |v|^2}, \tag{2.5}$$

$\xi_{10} = z_{10}/|z_{10}|$, and $\langle v, \xi_{10}\rangle$ is the scalar product of the vectors v and ξ_{10} in $\mathbb{R}^n$.

It is worth noting that $\gamma(v, z_{10})$ in (2.5) is ordinarily termed *a resolving function*.
Here are some elementary properties of (2.4) and (2.5).

P r o p o s i t i o n 2.2.

$$\alpha - \beta \leq \gamma(v, z_{10}) \leq \alpha + \beta \tag{2.6}$$

if and only if $\alpha \geq \beta$.

P r o p o s i t i o n 2.3. $|\boldsymbol{u}_\Pi(v, z_{10})| = \alpha$ *holds for any $v \in B_\beta$.*

D e f i n i t i o n 2.2. For an arbitrary control $v(\cdot) \in \mathbb{V}$, the scalar function

$$\Gamma(v(\cdot), t) = 1 - \frac{1}{a|z_{10}|}\int_0^t \gamma(v(s), z_{10})\left(1 - e^{-a(t-s)}\right) ds \tag{2.7}$$

is called *an approach function of the players X and Y* if $\alpha > \beta$.

L e m m a 2.1. *Let $\alpha > \beta$. Then for any control $v(\cdot) \in \mathbb{V}$:*
(a) $\Gamma(v(\cdot), t)$ *monotonically decreases with t, $t \geq 0$;*
(b) $\Gamma(v(\cdot), t)$ *at each $t \in \left[0, \widehat{T}\right]$ is estimated as*

$$\Gamma_1(t) \leq \Gamma(v(\cdot), t) \leq \Gamma_2(t), \tag{2.8}$$

where

$$\Gamma_1(t) = 1 - \frac{\alpha + \beta}{a^2|z_{10}|}\left(e^{-at} + at - 1\right),$$
$$\Gamma_2(t) = 1 - \frac{\alpha - \beta}{a^2|z_{10}|}\left(e^{-at} + at - 1\right).$$

P r o o f. a) Determine the t-derivative of $\Gamma(v(\cdot), t)$ and from (2.6) it follows that

$$\frac{d\Gamma(v(\cdot), t)}{dt} = -\frac{1}{|z_{10}|}\int_0^t \gamma(v(s), z_{10})e^{-a(t-s)}ds \leq \frac{\alpha - \beta}{a|z_{10}|}\left(e^{-at} - 1\right) \leq 0.$$

b) Relying on the lemma about minimum–maximum in [36], we make the following estimates:

$$\Gamma(v(\cdot),t) \le 1 - \frac{1}{a|z_{10}|}\min_{v(\cdot)\in\mathbb{V}}\int_0^t \gamma(v(s),z_{10})\big(1-e^{-a(t-s)}\big)\,ds \le$$
$$\le 1 - \frac{1}{a|z_{10}|}\Big(t-\frac{1}{a}\big(1-e^{-at}\big)\Big)\min_{v\in B_\beta}\gamma(v,z_{10}) = \Gamma_2(t).$$

By the same lemma, we obtain

$$\Gamma_1(t) = 1 - \frac{1}{a|z_{10}|}\Big(t-\frac{1}{a}\big(1-e^{-at}\big)\Big)\max_{v\in B_\beta}\gamma(v,z_{10}) =$$
$$= 1 - \frac{1}{a|z_{10}|}\max_{v(\cdot)\in\mathbb{V}}\int_0^t \gamma(v(s),z_{10})\big(1-e^{-a(t-s)}\big)ds \le \Gamma(v(\cdot),t),$$

and the proof is complete. □

We can now formulate our main result for the problem of pursuit.

T h e o r e m 2.1. *If* $\alpha > \beta$, *then the* Π-*strategy* (2.4) *will be winning for the pursuer on the time interval* $\left[0,\widehat{T}\right]$, *where* $\widehat{T}$ *is the positive root of* (2.3).

P r o o f. Let us first assume that the evader decides on any control $v(\cdot)\in\mathbb{V}$ and the pursuer utilizes the Π-strategy (2.4). Then by virtue of (2.1) and (2.4), we have

$$z(t) = z_{10} - \frac{\xi_{10}}{a}\int_0^t \big(1-e^{-a(t-s)}\big)\gamma(v(s),z_{10})\,ds. \tag{2.9}$$

Since $\xi_{10} = z_{10}/|z_{10}|$, rewrite (2.9) as

$$z(t) = \Gamma(v(\cdot),t)z_{10}, \tag{2.10}$$

where $\Gamma(v(\cdot),t)$ is the same as (2.7). Taking (2.8) into consideration yields

$$\Gamma(v(\cdot),t) \le \Gamma_2(t). \tag{2.11}$$

Using (2.2), reduce $\Gamma_2(t)$ to the form

$$\Gamma_2(t) = \frac{\beta-\alpha}{a^2|z_{10}|}p(t).$$

From Proposition 2.1 it follows immediately that at a moment $\widehat{T}$, the equality $p\left(\widehat{T}\right) = 0$ is satisfied. We thus get $\Gamma_2\left(\widehat{T}\right) = 0$ and, in consequence, (2.11) indicates that there exists a finite time $T_* \in \left[0,\widehat{T}\right]$ satisfying $\Gamma(v(\cdot),T_*) = 0$. For this reason, considering (2.10) we obtain $z(T_*) = 0$, which completes the proof. □

§ 3. Solution of the problem of evasion

In this section, a constant function will be taken as a strategy of the evader and a sufficient condition of evasion will be given. Moreover, we will substantiate that a strategy of the evader is an optimal strategy and the time $\widehat{T}$ stated in Theorem 2.1 is an optimal capture time.

D e f i n i t i o n 3.1. In the game $(\mathbb{U},\mathbb{V})$, the control function

$$\boldsymbol{v}_*(t) = -\beta\xi_{10} \tag{3.1}$$

is called *a strategy of the evader*, where $\xi_{10} = z_{10}/|z_{10}|$.

Let us state our main result for the problem of evasion.

T h e o r e m 3.1. (a) *If* $\alpha > \beta$, *then the strategy* (3.1) *will be winning for the evader in the time interval* $\Big[0, \widehat{T}\Big)$;

(b) *If* $\alpha \le \beta$, *then the strategy* (3.1) *will be winning for the evader in the time interval* $[0, +\infty)$ *and for the distance between the players* X *and* Y, *the following estimate is satisfied:*

$$|z(t)| \ge \begin{cases} |z_{10}|, & \text{if } \alpha = \beta, \\ |z_{10}| - \Big((\alpha - \beta)\big(e^{-at} + at - 1\big)/a^2\Big), & \text{if } \alpha < \beta. \end{cases}$$

P r o o f. (a) Let $\alpha > \beta$. Suppose that $u(\cdot) \in \mathbb{U}$ is an optional control of the pursuer and the evader applies the strategy (3.1). Then by means of (2.1), we have

$$z(t) = z_{10} + \frac{1}{a}\int_0^t u(s)\big(1 - e^{-a(t-s)}\big)\,ds + \frac{\beta\xi_{10}}{a}\int_0^t \big(1 - e^{-a(t-s)}\big)\,ds. \tag{3.2}$$

Taking (1.3) into account, the absolute value of (3.2) can be evaluated from below as follows:

$$|z(t)| \ge \left|z_{10} + \frac{\beta\xi_{10}}{a}\int_0^t \big(1 - e^{-a(t-s)}\big)\,ds\right| - \left|\frac{1}{a}\int_0^t u(s)\big(1 - e^{-a(t-s)}\big)\,ds\right| \ge$$
$$\ge |z_{10}|\left(1 + \frac{\beta}{a|z_{10}|}\left(t - \frac{1}{a}\big(1 - e^{-at}\big)\right)\right) - \frac{1}{a}\int_0^t |u(s)|\big(1 - e^{-a(t-s)}\big)\,ds =$$
$$= |z_{10}|\left(1 - \frac{\alpha - \beta}{a|z_{10}|}\left(t - \frac{1}{a}\big(1 - e^{-at}\big)\right)\right) = |z_{10}|\Gamma_2(t).$$

Based on the proof of Theorem 2.1, we can assert that the relation

$$|z(t)| \ge |z_{10}|\Gamma_2(t) > 0$$

is true for each $t \in \Big[0, \widehat{T}\Big)$.

(b) Let $\alpha \le \beta$. Then obtain the estimate as above once again and by (2.8), the following inequality is derived:

$$|z(t)| \ge |z_{10}| - \frac{\alpha - \beta}{a^2}\big(e^{-at} + at - 1\big) = \Gamma_3(t). \tag{3.3}$$

Due to $\Gamma_3(0) = |z_{10}|$ and $\dot{\Gamma}_3(t) \ge 0$ in the interval $[0, +\infty)$, it is evident that $\min_{t \ge 0}\Gamma_3(t) = |z_{10}|$. This leads to $z(t) \ne 0$, which follows from (3.3), for all t, $t \in [0, +\infty)$. The proof is complete.

§ 4. Attainability domain of players and its dynamics

In Theorem 2.1, we have proved that the pursuer seizes the evader through the Π-strategy (2.4) at some point in $\mathbb{R}^n$ by the time $\widehat{T}$ iff $\alpha > \beta$. Hence we are going to construct the set of capture points for the game $(\mathbb{U}, \mathbb{V})$.

The triplet $(y_{10}, y_{20}, v(\cdot))$, $v(\cdot) \in \mathbb{V}$, produces the trajectory of the evader in the form (1.6) and the triplet $(x_{10}, x_{20}, \boldsymbol{u}_\Pi(v(\cdot), z_{10}))$, $\boldsymbol{u}_\Pi(v(\cdot), z_{10}) \in \mathbb{U}$, yields the trajectory

$$x(t) = x_{10} + \frac{x_{20}}{a}\left(1 - e^{-at}\right) + \frac{1}{a}\int_0^t \boldsymbol{u}_\Pi(v(s), z_{10})\left(1 - e^{-a(t-s)}\right)ds \tag{4.1}$$

of the pursuer, where $t \in [0, T_*]$, $0 < T_* \le \widehat{T}$, and T_* is the encounter time of the players X and Y, i.e., $x(T_*) = y(T_*)$ holds. Accordingly, for each pair $(x(t), y(t))$ on $[0, T_*]$, we will consider the multi-valued mapping

$$\Omega(x(t), y(t)) = \big\{\omega\colon |\omega - x(t)| \ge (\alpha/\beta)|\omega - y(t)|\big\}, \tag{4.2}$$

and also, at $t = 0$ we have

$$\Omega(x_{10}, y_{10}) = \big\{\omega\colon |\omega - x_{10}| \geq (\alpha/\beta)|\omega - y_{10}|\big\}. \tag{4.3}$$

Note that $y(t) \in \Omega(x(t), y(t))$ for each $t \in [0, T_*]$ is valid owing to $|z(t)| \geq 0$ on the interval $[0, T_*]$.

R e m a r k 4.1. In fact, the trajectories $x(t)$, $y(t)$ and the set $\Omega(x(t), y(t))$ directly depend on how a control $v(\cdot) \in \mathbb{V}$ to be picked. For simplicity, we ignore this dependence.

L e m m a 4.1. *For the set* (4.2)*,*

$$\Omega(x(t), y(t)) = x(t) + \Gamma(v(\cdot), t)\big[\Omega(x_{10}, y_{10}) - x_{10}\big] \tag{4.4}$$

is satisfied, where $\Gamma(v(\cdot), t)$ *is identical with* (2.7) *and*

$$\Omega(x_{10}, y_{10}) = x_{10} - c(z_{10}) + r(z_{10})B, \tag{4.5}$$

$$c(z_{10}) = \frac{\alpha^2 z_{10}}{\alpha^2 - \beta^2}, \quad r(z_{10}) = \frac{\alpha\beta|z_{10}|}{\alpha^2 - \beta^2}, \tag{4.6}$$

and B *is the unit ball centred at the origin in* $\mathbb{R}^n$*.*

P r o o f. Since $z(t) = x(t) - y(t)$ (see (1.7)), it derives readily from (4.2) that the inclusion $\omega \in \Omega(x(t), y(t)) - x(t)$ is equivalent to

$$|\omega| \geq (\alpha/\beta)|\omega + z(t)|. \tag{4.7}$$

By reason of (4.2) and (4.7), the set $\Omega(t)$ can be represented as

$$\Omega(x(t), y(t)) = x(t) + \Omega_*(z(t)), \quad \Omega_*(z(t)) = \big\{\omega\colon |\omega| \geq (\alpha/\beta)|\omega + z(t)|\big\}. \tag{4.8}$$

Let us confirm that the set $\Omega_*(z(t))$ consists of a sphere. To do this, square both sides of (4.7) and make the following simplifications:

$$\beta^2|\omega|^2 \geq \alpha^2\big(|\omega|^2 + 2\langle\omega, z(t)\rangle + |z(t)|^2\big);$$
$$\big(\alpha^2 - \beta^2\big)\,|\omega|^2 + 2\alpha^2\langle\omega, z(t)\rangle + \alpha^2|z(t)|^2 \leq 0.$$

Divide both sides of the last inequality by $(\alpha^2 - \beta^2)$, i. e.,

$$|\omega|^2 + 2\left\langle\omega,\ \frac{\alpha^2 z(t)}{\alpha^2 - \beta^2}\right\rangle + \frac{\alpha^2|z(t)|^2}{\alpha^2 - \beta^2} \leq 0. \tag{4.9}$$

Adding $\Big(\alpha^2 z(t)/\left(\alpha^2 - \beta^2\right)\Big)^2$ to both sides of (4.9), we attain

$$|\omega|^2 + 2\left\langle\omega,\ \frac{\alpha^2 z(t)}{\alpha^2 - \beta^2}\right\rangle + \left(\frac{\alpha^2 z(t)}{\alpha^2 - \beta^2}\right)^2 \leq \left(\frac{\alpha^2 z(t)}{\alpha^2 - \beta^2}\right)^2 - \frac{\alpha^2|z(t)|^2}{\alpha^2 - \beta^2} =$$
$$= \left(\frac{\alpha^4}{(\alpha^2 - \beta^2)^2} - \frac{\alpha^2}{\alpha^2 - \beta^2}\right)|z(t)|^2 = \left(\frac{\alpha\beta|z(t)|}{\alpha^2 - \beta^2}\right)^2,$$

or

$$\left|\omega + \frac{\alpha^2 z(t)}{\alpha^2 - \beta^2}\right| \leq \frac{\alpha\beta|z(t)|}{\alpha^2 - \beta^2}. \tag{4.10}$$

So, the relation (4.10) is a sphere, which center is at the point $c(z(t)) = -\alpha^2 z(t)/\left(\alpha^2-\beta^2\right)$ and which radius is equal to $r(z(t)) = \alpha\beta|z(t)|/\left(\alpha^2-\beta^2\right)$, thereby describing $\Omega_*(z(t))$ in (4.8) as

$$\Omega_*(z(t)) = \left\{\omega\colon |\omega - c(z(t))| \le r(z(t))\right\}. \tag{4.11}$$

Moreover, the set (4.11) can be written in the form $\Omega_*(z(t)) = c(z(t)) + r(z(t))B$ (see [37, p. 24]), and on that account it proceeds from (4.8) that

$$\Omega(x(t), y(t)) = x(t) + \Omega_*(z(t)) = x(t) + c(z(t)) + r(z(t))B. \tag{4.12}$$

By (2.10) and (4.6), the functions $c(z(t))$ and $r(z(t))$ convert to the forms

$$c(z(t)) = -c(z_{10})\Gamma(v(\cdot), t), \quad r(z(t)) = r(z_{10})\Gamma(v(\cdot), t). \tag{4.13}$$

Replacing (4.13) into (4.12) gives (4.4), and the proof is complete. □

C o r o l l a r y 4.1. *From Lemma* 4.1 *it may be concluded that at each* $t \in [0, T_*]$, *the set* $\Omega(x(t), y(t))$ *is a ball of the radius* $r(z_{10})\Gamma(v(\cdot), t)$ *centered at the point* $x(t) - c(z_{10})\Gamma(v(\cdot), t)$ *and the set* $\Omega(x_{10}, y_{10})$ *is a ball of the radius* $r(z_{10})$ *centered at the point* $x_{10} - c(z_{10})$.

L e m m a 4.2 (main lemma). *Let*

$$\Psi\left(x(t), y(t), x_{20}\right) = \Omega(x(t), y(t)) - \frac{x_{20}}{a}\left(1 - e^{-at}\right).$$

Then $\Psi(x(t_2), y(t_2), x_{20}) \subset \Psi(x(t_1), y(t_1), x_{20})$ *is true when* $t_1 < t_2$ *for any* t_1, $t_2 \in [0, T_*]$.

P r o o f. It is not difficult to claim that (1.4) is equivalent to

$$|v(t)|^2 \le \frac{\beta^2}{\alpha^2-\beta^2}\left(\alpha^2 - |v(t)|^2\right). \tag{4.14}$$

From (2.5) we easily find

$$\alpha^2 - |v(t)|^2 = \gamma(v(t), z_{10})\left(\gamma(v(t), z_{10}) - 2\langle v(t), \xi_{10}\rangle\right). \tag{4.15}$$

Combining (4.15) with (4.14) and removing the brackets we get

$$|v(t)|^2 + \frac{2\beta^2\langle v(t), \xi_{10}\rangle}{\alpha^2-\beta^2}\gamma(v(t), z_{10}) \le \frac{\beta^2}{\alpha^2-\beta^2}\gamma^2(v(t), z_{10}). \tag{4.16}$$

Add $\left(\beta^2\gamma(v(t), z_{10})\xi_{10}/\left(\alpha^2-\beta^2\right)\right)^2$ to both sides of (4.16) and simplify the right side of the result, i. e.,

$$|v(t)|^2 + \frac{2\beta^2\langle v(t), \xi_{10}\rangle}{\alpha^2-\beta^2}\gamma(v(t), z_{10}) + \left(\frac{\beta^2\gamma(v(t), z_{10})\xi_{10}}{\alpha^2-\beta^2}\right)^2 \le \frac{\beta^2}{\alpha^2-\beta^2}\gamma^2(v(t), z_{10}) +$$
$$+ \left(\frac{\beta^2\gamma(v(t), z_{10})\xi_{10}}{\alpha^2-\beta^2}\right)^2 = \left(\frac{\beta^2}{\alpha^2-\beta^2} + \frac{\beta^4}{\left(\alpha^2-\beta^2\right)^2}\right)\gamma^2(v(t), z_{10}) = \left(\frac{\alpha\beta}{\alpha^2-\beta^2}\gamma(v(t), z_{10})\right)^2,$$

or

$$\left|v(t) + \frac{\beta^2}{\alpha^2-\beta^2}\gamma(v(t), z_{10})\xi_{10}\right| \le \frac{\alpha\beta}{\alpha^2-\beta^2}\gamma(v(t), z_{10}). \tag{4.17}$$

Clearly, the inequality

$$\left\langle v(t) + \frac{\beta^2}{\alpha^2-\beta^2}\gamma(v(t), z_{10})\xi_{10},\ \psi\right\rangle \le \left| v(t) + \frac{\beta^2}{\alpha^2-\beta^2}\gamma(v(t), z_{10})\xi_{10}\right|$$

is valid for an arbitrary vector $\psi \in \mathbb{R}^n$, $|\psi| = 1$. From this and (4.17) we determine

$$\left\langle v(t) + \frac{\beta^2}{\alpha^2-\beta^2}\gamma(v(t), z_{10})\xi_{10},\ \psi\right\rangle \le \frac{\alpha\beta}{\alpha^2-\beta^2}\gamma(v(t), z_{10}). \tag{4.18}$$

Multiplying (4.18) by $e^{-a(t-s)}$, $0 \le s \le t$, we take

$$\left\langle \left(v(t) + \frac{\beta^2}{\alpha^2-\beta^2}\gamma(v(t), z_{10})\xi_{10}\right) e^{-a(t-s)},\ \psi\right\rangle \le \frac{\alpha\beta e^{-a(t-s)}}{\alpha^2-\beta^2}\gamma(v(t), z_{10}). \tag{4.19}$$

Integrate (4.19) on $[0, t]$ and obtain that

$$\int_0^t \left\langle \left(v(s) + \frac{\beta^2}{\alpha^2-\beta^2}\gamma(v(s), z_{10})\xi_{10}\right) e^{-a(t-s)},\ \psi\right\rangle ds \le \frac{\alpha\beta}{\alpha^2-\beta^2}\int_0^t \gamma(v(s), z_{10})e^{-a(t-s)}ds. \tag{4.20}$$

From the left side of (4.20) we get the following equalities:

$$\int_0^t \left\langle \left(v(s) + \frac{\beta^2}{\alpha^2-\beta^2}\gamma(v(s), z_{10})\xi_{10}\right) e^{-a(t-s)},\ \psi\right\rangle ds =$$
$$= \int_0^t \left\langle \left(v(s) + \left(\frac{\alpha^2}{\alpha^2-\beta^2} - 1\right)\gamma(v(s), z_{10})\xi_{10}\right) e^{-a(t-s)},\ \psi\right\rangle ds =$$
$$= \int_0^t \Big\langle \big(v(s) - \gamma(v(s), z_{10})\xi_{10}\big)e^{-a(t-s)},\ \psi\Big\rangle ds + \Big\langle \frac{\alpha^2}{\alpha^2-\beta^2}\xi_{10},\ \psi\Big\rangle \int_0^t \gamma(v(s), z_{10})e^{-a(t-s)}\, ds.$$

In view of the Π-strategy (2.4) and the vector $c(z_{10})$ in (4.6), from the last equality we obtain

$$\begin{aligned}&\int_0^t \left\langle \left(v(s) + \frac{\beta^2}{\alpha^2-\beta^2}\gamma(v(s), z_{10})\xi_{10}\right) e^{-a(t-s)},\ \psi\right\rangle ds =\\ &= \left\langle \int_0^t \boldsymbol{u}_\Pi(v(s), z_{10})e^{-a(t-s)}ds,\ \psi\right\rangle + \frac{\langle c(z_{10}), \psi\rangle}{|z_{10}|}\int_0^t \gamma(v(s), z_{10})e^{-a(t-s)}\, ds.\end{aligned} \tag{4.21}$$

By virtue of $r(z_{10})$ in (4.6), the right side of (4.20) can be expressed as

$$\frac{\alpha\beta}{\alpha^2-\beta^2}\int_0^t \gamma(v(s), z_{10})e^{-a(t-s)}\, ds = \frac{r(z_{10})}{|z_{10}|}\int_0^t \gamma(v(s), z_{10})e^{-a(t-s)}\, ds. \tag{4.22}$$

As a consequence of (4.20)–(4.22), we deduce that

$$\left\langle \int_0^t \boldsymbol{u}_\Pi(v(s), z_{10})e^{-a(t-s)}ds,\ \psi\right\rangle + \frac{\langle c(z_{10}), \psi\rangle - r(z_{10})}{|z_{10}|}\int_0^t \gamma(v(s), z_{10})e^{-a(t-s)}\, ds \le 0. \tag{4.23}$$

Since, here, $\Omega(x(t), y(t))$ is mainly considered as the ball with a center and radius changing in time, it is easy to calculate its support function $F\big(\Omega(x(t), y(t)),\ \psi\big)$ for any vector $\psi \in \mathbb{R}^n$, $|\psi| = 1$ (see [37]). Then taking account of (2.7), (4.1), (4.4)–(4.6) and on the strength of the properties of a support function, we take the t-derivative of $F\big(\Omega(x(t), y(t)),\ \psi\big)$, i.e.,

$$\frac{d}{dt}F\big(\Omega(x(t), y(t)),\ \psi\big) = \frac{d}{dt}F\big(x(t) + \Gamma(v(\cdot), t)\big[r(z_{10})B - c(z_{10})\big],\ \psi\big) =$$
$$= \frac{d}{dt}F\big(x(t), \psi\big) + F\big(r(z_{10})B - c(z_{10}),\ \psi\big)\frac{d}{dt}\Gamma(v(\cdot), t) =$$

$$= \frac{d}{dt}\left\langle x_{10} + \frac{x_{20}}{a}\big(1-e^{-at}\big) + \frac{1}{a}\int_0^t \boldsymbol{u}_\Pi(v(s), z_{10})\big(1-e^{-a(t-s)}\big)ds,\ \psi\right\rangle +$$
$$+ \big(r(z_{10})F(B,\psi) - F(c(z_{10}),\psi)\big)\frac{d}{dt}\left(1 - \frac{1}{a|z_{10}|}\int_0^t \gamma(v(s), z_{10})\big(1-e^{-a(t-s)}\big)ds\right).$$

As $F(B,\psi) = 1$ and $F(c(z_{10}),\psi) = \langle c(z_{10}),\psi\rangle$ (see [37, p. 33]), from the latest equalities we arrive at the relation

$$\frac{d}{dt}F\big(\Omega(x(t),y(t)),\ \psi\big) = \big\langle e^{-at}x_{20},\psi\big\rangle + \left\langle\int_0^t \boldsymbol{u}_\Pi(v(s), z_{10})e^{-a(t-s)}ds,\ \psi\right\rangle + \\ + \frac{\langle c(z_{10}),\psi\rangle - r(z_{10})}{|z_{10}|}\int_0^t \gamma(v(s), z_{10})e^{-a(t-s)}\,ds. \tag{4.24}$$

Accommodating (4.24) we attain the following:

$$\frac{d}{dt}F\big(\Omega(x(t),y(t)) - \frac{x_{20}}{a}\big(1-e^{-at}\big),\ \psi\big) =$$
$$= \frac{d}{dt}F\big(\Omega(x(t),y(t)),\ \psi\big) - \frac{d}{dt}\left(\frac{x_{20}}{a}\big(1-e^{-at}\big),\ \psi\right) =$$
$$= \big\langle e^{-at}x_{20},\psi\big\rangle + \left\langle\int_0^t \boldsymbol{u}_\Pi(v(s), z_{10})e^{-a(t-s)}\,ds,\ \psi\right\rangle +$$
$$+ \frac{\langle c(z_{10}),\psi\rangle - r(z_{10})}{|z_{10}|}\int_0^t \gamma(v(s), z_{10})e^{-a(t-s)}\,ds - \big\langle e^{-at}x_{20},\psi\big\rangle,$$

or

$$\frac{d}{dt}F\big(\Omega(t) - \frac{x_{20}}{a}\big(1-e^{-at}\big),\ \psi\big) =$$
$$= \left\langle\int_0^t \boldsymbol{u}_\Pi(v(s), z_{10})e^{-a(t-s)}\,ds,\ \psi\right\rangle + \frac{\langle c(z_{10}),\psi\rangle - r(z_{10})}{|z_{10}|}\int_0^t \gamma(v(s), z_{10})e^{-a(t-s)}\,ds.$$

As a result of (4.23), we see that

$$\frac{d}{dt}F\big(\Omega(x(t),y(t)) - \frac{x_{20}}{a}\big(1-e^{-at}\big),\ \psi\big) \le 0$$

holds for every $\psi \in \mathbb{R}^n$, $|\psi| = 1$. This finishes the proof. □

By Lemma 4.2, we will construct an attainability domain of the evader.

L e m m a 4.3. *The inclusion*

$$y(t) \in \Omega(x_{10}, y_{10}) + \frac{x_{20}}{a}\big(1-e^{-at}\big) \tag{4.25}$$

holds for all $t \in [0, T_*]$.

P r o o f. On account of Lemma 4.2, we derive the relation

$$\Omega(x(t),y(t)) - \frac{x_{20}}{a}\big(1-e^{-at}\big) \subset \Omega(x_{10}, y_{10}),$$

or

$$\Omega(x(t),y(t)) \subset \Omega(x_{10}, y_{10}) + \frac{x_{20}}{a}\big(1-e^{-at}\big). \tag{4.26}$$

Obviously, $y(t) \in \Omega(x(t),y(t))$ on the interval $[0, T_*]$ (see (4.2)) and therefore from (4.26) it proceeds the validity of (4.25), which completes the proof. □

D e f i n i t i o n 4.1. We call the set

$$\Omega_X\left(x_{10}, x_{20}, y_{10}, \widehat{T}\right) = \bigcup_{t=0}^{\widehat{T}} \left\{\Omega(x_{10}, y_{10}) + \frac{x_{20}}{a}\left(1 - e^{-at}\right)\right\}$$

the attainability domain of the pursuer in the game $(\mathbb{U}, \mathbb{V})$, where $\widehat{T}$ is the guaranteed capture time (see Theorem 2.1).

C o r o l l a r y 4.2. *If* $x_{20} = y_{20} = 0$, *then the attainability domain of the evader consists of the set* $\Omega(x_{10}, y_{10})$ *(the Apollonius sphere) in* (4.3).

§ 5. Solution of the “lifeline” game

In the present section, the game (1.1)–(1.4) with the “lifeline” L will be solved to the advantage of both players X and Y.

D e f i n i t i o n 5.1. In the “lifeline” game, the Π-strategy (2.4) is said to be *winning for the pursuer* on the time interval $\left[0, \widehat{T}\right]$ if, for an arbitrary control $v(\cdot) \in \mathbb{V}$ of the evader, there is some moment $\hat{t} \in \left[0, \widehat{T}\right]$ such that:
(1) $x\left(\hat{t}\right) = y\left(\hat{t}\right)$;
(2) $y(\tau) \notin L$ at each $\tau \in [0, \hat{t}]$.

T h e o r e m 5.1. *If* $\alpha > \beta$ *and* $\Omega_X\left(x_{10}, x_{20}, y_{10}, \widehat{T}\right) \cap L = \emptyset$, *then the* Π*-strategy* (2.4) *will be winning for the pursuer on the time interval* $\left[0, \widehat{T}\right]$ *in the “lifeline” game.*

P r o o f. The proof follows immediately from Theorem 2.1, Lemma 4.2 and 4.3. □

And now, we will consider the “lifeline” game from the point of view of the evader.

D e f i n i t i o n 5.2. It is said that a control $v_\star(\cdot) \in \mathbb{V}$ is *winning for the evader* in the “lifeline” game if, for an optional control $u(\cdot) \in \mathbb{U}$ of the pursuer:
(a) there is some finite moment $\tilde{t}$ such that $y\left(\tilde{t}\right) \in L$ and $x(t) \neq y(t)$ for $0 \leq t < \tilde{t}$; or
(b) $x(t) \neq y(t)$ at each $t \in [0, +\infty)$.

Let us take the sets

$$T_\star(\omega, y_{10}) = \left\{t_\star \colon e^{-at_\star} + at_\star = 1 + \frac{a^2|\omega - y_{10}|}{\beta},\ \omega \in \Omega(0)\right\}, \tag{5.1}$$

$$\Omega_Y(\omega, x_{20}) = \left\{\omega_\star \colon \omega_\star = \omega + \frac{x_{20}}{a}\left(1 - e^{-at_\star}\right),\ \omega \in \Omega(0)\right\}. \tag{5.2}$$

In the “lifeline” game, we call the set (5.2) *the attainability domain of the evader*.

T h e o r e m 5.2. *If* $\alpha > \beta$ *and* $\Omega_Y(\omega, x_{20}) \cap L \neq \emptyset$, *then there is such control* $v_\star(\cdot) \in \mathbb{V}$ *that is winning for the evader in the “lifeline” game.*

P r o o f. According to the theorem, there exists at least one point $\omega_\star \in \Omega_Y(\omega, x_{20}) \cap L$ satisfying the equality in (5.2). Then we fix the evader up with the constant control

$$v_\star = \beta\frac{\omega - y_{10}}{|\omega - y_{10}|}. \tag{5.3}$$

First off, let us demonstrate that by the control (5.3) the evader gets to the target point $\omega_\star$ at some time $t_\star \in T_\star(\omega, y_{10})$. Indeed, due to $x_{20} = y_{20}$, by means of (1.6) we obtain

$$y\left(t_\star\right) = y_{10} + \frac{x_{20}}{a}\left(1 - e^{-at_\star}\right) + \frac{v_\star}{a^2}\left(e^{-at_\star} + at_\star - 1\right). \tag{5.4}$$

Substituting (5.3) in (5.4) and allowing for the equality in (5.1) we achieve the desired result, i. e.,

$$y\left(t_\star\right)=y_{10}+\frac{x_{20}}{a}\left(1-e^{-at_\star}\right)+\omega-y_{10}=\omega_\star.$$

Now, we need to show the validity of the condition (a) of Definition 5.2, that is, the evader can escape by using (5.3) on the interval $[0,t_\star]$. Assume conversely, or, more precisely, for the pursuer there exists a control $u_\star(\cdot)\in\mathbb{U}$ by which the equality $x\left(t^\star\right)=y\left(t^\star\right)$ occurs at some $t^\star\in[0,t_\star)$. Consequently, depending on (2.1), we get the vector

$$z\left(t^\star\right)=z_{10}+\frac{1}{a}\int_0^{t^\star}\left(u_\star(s)-v_\star\right)\left(1-e^{-a(t^\star-s)}\right)ds. \tag{5.5}$$

It is easy to see that $z\left(t^\star\right)=0$ holds, thanks to $z(t)=x(t)-y(t)$ (see (1.7)). As a result, considering (1.3) we find the following estimations from the right side of (5.5):

$$\left|z_{10}-\frac{1}{a}\int_0^{t^\star}\left(1-e^{-a(t^\star-s)}\right)v_\star\,ds\right|\le\frac{1}{a}\int_0^{t^\star}\left(1-e^{-a(t^\star-s)}\right)|u_\star(s)|\,ds\le\frac{\alpha}{a^2}\left(e^{-at^\star}+at^\star-1\right),$$

or

$$\left|z_{10}-\frac{v_\star}{a^2}\left(e^{-at^\star}+at^\star-1\right)\right|\le\frac{\alpha}{a^2}\left(e^{-at^\star}+at^\star-1\right). \tag{5.6}$$

Let us introduce the notation $\lambda=\left(e^{-at^\star}+at^\star-1\right)/a^2$ in (5.6). Then square both sides of (5.6) and because of $|v_\star|=\beta$, we come to the quadratic inequality with λ of the form

$$\left(\alpha^2-\beta^2\right)\lambda^2+2\langle z_{10},v_\star\rangle\lambda-|z_{10}|^2\ge0.$$

As $\lambda>0$, from the latter inequality we derive

$$\frac{1}{\alpha^2-\beta^2}\left(\sqrt{\langle z_{10},v_\star\rangle^2+(\alpha^2-\beta^2)\,|z_{10}|^2}-\langle z_{10},v_\star\rangle\right)\le\lambda=\frac{1}{a^2}\left(e^{-at^\star}+at^\star-1\right). \tag{5.7}$$

Owing to $t^\star<t_\star$ and taking the equality in (5.1) on board, it arises from (5.7) that

$$\frac{1}{\alpha^2-\beta^2}\left(\sqrt{\langle z_{10},v_\star\rangle^2+(\alpha^2-\beta^2)\,|z_{10}|^2}-\langle z_{10},v_\star\rangle\right)<\frac{|w-y_{10}|}{\beta}. \tag{5.8}$$

In accordance with (5.3), we find by (5.8) that $\beta|\omega-x_{10}|<\alpha|\omega-y_{10}|$, i. e., $\omega\notin\Omega(x_{10},y_{10})$ (see (4.3)), and this conflicts with our supposition. The proof is complete. □

R e m a r k 5.1. According to the definitions of $\Omega_X\left(x_{10},x_{20},y_{10},\widehat{T}\right)$ and $\Omega_Y(\omega,x_{20})$, we can conclude that $\Omega_Y(\omega,x_{20})\subset\Omega_X\left(x_{10},x_{20},y_{10},\widehat{T}\right)$ is reasonable in the "lifeline" game.

T h e o r e m 5.3. *Let* $\alpha\le\beta$. *Then there is a control* $v_\star(\cdot)\in\mathbb{V}$, *which is winning for the evader in the "lifeline" game.*

P r o o f. The proof follows directly from Theorem 3.1. □

§ 6. Examples

Below we will first address a problem for the game with the "lifeline".

E x a m p l e 6.1. Let us analyze the game given by the following differential equations and initial conditions:

$$X\colon \quad \ddot{x}+\dot{x}=u, \quad x_{10}=(7,14), \quad x_{20}=(0,1), \quad |u(t)|\le 2, \quad t\ge 0, \tag{6.1}$$
$$Y\colon \quad \ddot{y}+\dot{y}=v, \quad y_{10}=(4,8), \quad y_{20}=(0,1), \quad |v(t)|\le 1, \quad t\ge 0. \tag{6.2}$$

In terms of (2.2) and Proposition 2.1, we get $\widehat{T}\approx 7.71$. On the basis of (4.5) and (4.6), we obtain

$$\Omega(x_{10},y_{10})=\big\{\omega=(\omega_1,\omega_2)\colon (\omega_1-4)^2+(\omega_2-8)^2\le 20\big\}. \tag{6.3}$$

It is obvious that the set (6.3) is equivalent to

$$\Omega(r)=\Big\{\omega_r=\Big(\omega_r^1,\omega_r^2\Big)\colon \ \big(\omega_r^1-4\big)^2+\big(\omega_r^2-8\big)^2=r^2,\ r\in\big[0,2\sqrt{5}\,\big]\Big\}. \tag{6.4}$$

Relying on (6.4) we determine the set (5.1) as follows:

$$T_\star(r)=\Big\{t_r\colon e^{-t_r}+t_r=1+r,\ r\in\!\big(0,2\sqrt{5}\,\big]\Big\}.$$

In view of (5.2) and (6.4), we generate the system

$$\begin{cases}\omega_\star^1=\omega_r^1,\\ \omega_\star^2=\omega_r^2+1-e^{-t_r}\end{cases}$$

for each $r\in\big(0,2\sqrt{5}\,\big]$. From this system we find the set

$$\Omega_Y(r,t_r)=\Big\{\omega_\star=\big(\omega_\star^1,\omega_\star^2\big):\omega_\star^2=8\pm\sqrt{r^2-(\omega_\star^1-4)^2}+1-e^{-t_r},\ r\in\!\Big(0,2\sqrt{5}\,\Big]\Big\}$$

as the attainability domain of the evader in the game (6.1), (6.2) with the "lifeline".

Now, we will consider constructing the attainability domain for the case with multiple pursuers and one evader.

E x a m p l e 6.2. Let the following game example be given:

$$X_i\colon \quad \ddot{x}_i+a\dot{x}_i=u_i, \quad x_i(0)=x_{i0}, \quad \dot{x}_i(0)=\theta, \quad |u_i(t)|\le \delta_i, \quad t\ge 0, \tag{6.5}$$
$$Y\colon \quad \ddot{y}+a\dot{y}=v, \quad y(0)=y_0, \quad \dot{y}(0)=\theta, \quad |v(t)|\le 1, \quad t\ge 0, \tag{6.6}$$

where $a>0$, $\delta_i>1$, $x_{i0}\ne y_0$, $i=\overline{1,m}$.

Based on Lemma 4.3, we can write the relation

$$y(t)\in\bigcap_{i=1}^m\Omega_i(x_{i0},y_0)+\frac{\theta}{a}\big(1-e^{-at}\big),$$

where

$$\Omega_i(x_{i0},y_0)=x_{i0}-c_i(z_{i0})+r_i(z_{i0})B,\quad c_i(z_{i0})=\delta_i^2 z_{i0}/\big(\delta_i^2-1\big),$$
$$r_i(z_{i0})=\delta_i|z_{i0}|/\big(\delta_i^2-1\big),\quad z_{i0}=x_{i0}-y_0.$$

For the game (6.5), (6.6) we can define the attainability domain of the pursuers in the form

$$\Omega_{X_1,\ldots,X_m}\big(x_{i0},y_0,\theta,\widetilde{T}\,\big)=\bigcup_{t=0}^{\widetilde{T}}\left[\,\bigcap_{i=1}^m\Omega_i(x_{i0},y_0)+\frac{\theta}{a}\big(1-e^{-at}\big)\right],$$

where

$$\widetilde{T}=\min_{i=\overline{1,m}}\left\{t\colon e^{-at}+at=1+\frac{a^2|z_{i0}|}{\delta_i-1}\right\}.$$

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Bahrom Tadjiahmatovich Samatov, Doctor of Physics and Mathematics, Professor, Department of Mathematical Analysis, Namangan State University, ul. Uychi, 316, Namangan, 116019, Uzbekistan.
ORCID: https://orcid.org/0000-0002-0734-8507
E-mail: samatov57@inbox.ru

Ulmasjon Boykuzi ugli Soyibboev, PhD student, Department of Mathematical Analysis, Namangan State University, ul. Uychi, 316, Namangan, 116019, Uzbekistan.
ORCID: https://orcid.org/0000-0003-0546-427X
E-mail: ulmasjonsoyibboev@gmail.com

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Б. Т. Саматов, У. Б. Сойиббоев

Дифференциальная игра с «линией жизни» для контрольного примера Понтрягина

Ключевые слова: дифференциальная игра, преследование, уклонение, ускорение, стратегия, гарантированное время поимки, область достижимости, линия жизни.

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Основной целью данной работы является решение одной из основных задач Айзекса, а именно игры с «линией жизни» на контрольном примере Понтрягина, когда оба игрока имеют одинаковую динамику движения. Для решения этой задачи преследователю предлагается стратегия параллельного преследования (кратко П-стратегия), обеспечивающая скорейшее сближение игроков и поимку убегающего в пределах некоторого замкнутого шара. Кроме того, для рассматриваемой дифференциальной игры приводится явная аналитическая формула для области достижимости игроков и обобщается основная лемма (лемма Л. А. Петросяна о монотонности относительно вложения области достижимости для игры простого преследования). Используя эту основную лемму, мы находим условия разрешимости игры с «линией жизни» и для контрольного примера Понтрягина. Для наглядности в конце работы приведены примеры для некоторых частных случаев.

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Саматов Бахром Таджиахматович, д. ф.-м. н., профессор, кафедра математического анализа, Наманганский государственный университет, 116019, Узбекистан, г. Наманган, ул. Уйчи, 316.
ORCID: https://orcid.org/0000-0002-0734-8507
E-mail: samatov57@inbox.ru

Сойиббоев Улмасжон Бойкузи угли, аспирант, кафедра математического анализа, Наманганский государственный университет, 116019, Узбекистан, г. Наманган, ул. Уйчи, 316.
ORCID: https://orcid.org/0000-0003-0546-427X
E-mail: ulmasjonsoyibboev@gmail.com

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