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INTEGRATION OF THE MKDV EQUATION WITH NONSTATIONARY COEFFICIENTS AND ADDITIONAL TERMS IN THE CASE OF MOVING EIGENVALUES

In this paper, we consider the Cauchy problem for the non-stationary modified Korteweg–de Vries equation with an additional term and a self-consistent source in the case of moving eigenvalues. Also, the evolution of the scattering data of the Dirac operator is obtained, the potential of which is the solution of the loaded modified Korteweg–de Vries equation with a self-consistent source in the class of rapidly decreasing functions. Specific examples are given to illustrate the application of the obtained results.

Keywords: Gelfand–Levitan–Marchenko integral equation, system of Dirac equations, Jost solutions, scattering data.

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Introduction

When integrating nonlinear evolutionary equations of mathematical physics, the main and difficult problem is to obtain exact solutions to nonlinear equations, including nonlinear wave, soliton, etc. Over the past few decades, many efficient methods have been developed to find such solutions for many integrable equations, such as the Korteweg–de Vries (KdV) equation and its various generalizations, various types of nonlinear Schrödinger equations, etc.

One of such integrable nonlinear equations is the following modified Korteweg–de Vries (mKdV) equation [1]:

$$u_t \pm 6u^2 u_x + u_{xxx} = 0.$$

This equation can be applied in many areas, such as the propagation of ultrashort solitons with a small number of optical cycles in nonlinear media [2, 3], anharmonic lattices [4], Alfven waves [5], ion-acoustic solitons [6–8], lines transmission through the Schottky barrier [9], thin oceanic jets [10, 11], internal waves [12, 13], thermal impulses in solids [14], etc. To calculate the exact solutions of the mKdV equation, many methods have been created, for example, the bilinear approach of Hirota [15], the Wronskian technique [16–18] can be mentioned. There are also a lot of results about the mKdV equation [19–25] due to its simple expression and rich physical application. In recent works by Vanneeva [26], one can find exact solutions of the mKdV equation with variable coefficients

$$u_t + u^2 u_x - g(t)u_{xxx} + h(t)u = 0.$$

In this paper, we consider the following system of equations

$$\begin{gathered} u_t + p(t)(6u^2 u_x + u_{xxx}) + q(t)u_x = \sum_{k=1}^{2N} \alpha_k(t)\big(f_{k1}g_{k1} - f_{k2}g_{k2}\big), \\ L(t)f_k = \xi_k f_k, \quad L(t)g_k = \xi_k g_k, \quad k = 1, 2, \ldots, 2N, \end{gathered} \tag{0.1}$$

where $L(t) = i\begin{pmatrix} \frac{d}{dx} & -u(x,t) \\ -u(x,t) & -\frac{d}{dx} \end{pmatrix}$ and $p(t)$, $q(t)$, $\alpha_k(t)$ $(k = \overline{1, 2N})$, are given continuously differentiable functions.

The system of equations (0.1) is considered under the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{0.2}$$

where the initial function $u_0(x)$ $(-\infty < x < \infty)$ has the following properties:

1)

$$\int_{-\infty}^{\infty} (1+|x|)|u_0(x)|\,dx < \infty; \tag{0.3}$$

2) the operator $L(0) = i\begin{pmatrix} \frac{d}{dx} & -u_0(x) \\ -u_0(x) & -\frac{d}{dx} \end{pmatrix}$ has exactly $2N$ simple eigenvalues $\xi_1(0)$, $\xi_2(0)$, $\ldots$, $\xi_{2N}(0)$.

In the problem under consideration, $f_k = \left(f_{k1}, f_{k2}\right)^T$ is the eigenfunction of the operator $L(t)$ corresponding to the eigenvalue ξ_k, and $g_k = \left(g_{k1}, g_{k2}\right)^T$ is the solution of the equation $Lg_k = \xi_k g_k$, for which

$$W\{f_k, g_k\} \equiv f_{k1}g_{k2} - f_{k2}g_{k1} = \omega_k(t) \neq 0, \qquad k = \overline{1, 2N},$$

where $\omega_k(t)$ are the initially given continuous functions of t, satisfying the conditions

$$\omega_n(t) = -\omega_k(t) \text{ in } \xi_n = -\xi_k, \quad \operatorname{Re}\left\{\int_0^t \omega_k(\tau)\,d\tau\right\} > -\operatorname{Im}\left\{\xi_k(0)\right\}, \quad k = \overline{1, N}, \tag{0.4}$$

for all non-negative values of t. For definiteness, we assume that in the sum in the right-hand side of (0.1), the terms with $\operatorname{Im}\xi_k > 0$, $k = \overline{1, N}$, come first.

Let us assume that the function $u(x,t)$ has the required smoothness and rather quickly tends to its limits at $x \to \pm\infty$, i. e.,

$$\int_{-\infty}^{+\infty} \left((1+|x|)|u(x,t)| + \sum_{k=1}^{3} \frac{\partial^k u(x,t)}{\partial x^k}\right) dx < \infty, \quad k = 1, 2, 3. \tag{0.5}$$

The main purpose of this work is to obtain representations for the solution $u(x,t)$, $f_k(x,t)$, $g_k(x,t)$, $k = \overline{1, 2N}$, of problem (0.1)–(0.5) in the framework of the inverse scattering method for the operator $L(t)$.

§ 1. Preliminaries

Consider the following system of Dirac equations

$$\begin{cases} v_{1x} + i\xi v_1 = u_0(x)v_2, \\ v_{2x} - i\xi v_2 = -u_0(x)v_1, \end{cases} \tag{1.1}$$

on the entire axis $(-\infty < x < \infty)$, with the potential $u_0(x)$ satisfying the condition (0.3).

It can be seen that using operator $L(0)$ and the vector function $\nu = (\nu_1, \nu_2)$, the system (1.1) can be rewritten as

$$L\nu = \xi\nu. \tag{1.2}$$

The system of equations (1.1) has Jost solutions with the following asymptotics

$$\left.\begin{aligned} \varphi(x,\xi) &\sim \begin{pmatrix}1\\0\end{pmatrix} e^{-i\xi x} \\ \widehat{\varphi}(x,\xi) &\sim \begin{pmatrix}0\\-1\end{pmatrix} e^{i\xi x} \end{aligned}\right\}, \quad \operatorname{Im}\xi = 0, \quad x \to -\infty;$$
$$\left.\begin{aligned} \psi(x,\xi) &\sim \begin{pmatrix}0\\1\end{pmatrix} e^{i\xi x} \\ \widehat{\psi}(x,\xi) &\sim \begin{pmatrix}1\\0\end{pmatrix} e^{-i\xi x} \end{aligned}\right\}, \quad \operatorname{Im}\xi = 0, \quad x \to \infty. \tag{1.3}$$

For real ξ, pairs of vector functions $\{\varphi, \widehat{\varphi}\}$ and $\{\psi, \widehat{\psi}\}$ are pairs of linearly independent solutions to the system of equations (1.1). Therefore, the following relations take place

$$\begin{cases} \varphi = a(\xi)\widehat{\psi} + b(\xi)\psi, \\ \widehat{\varphi} = -\bar{a}(\xi)\psi + \bar{b}(\xi)\widehat{\psi}, \end{cases} \quad \text{and} \quad \begin{cases} \psi = -a(\xi)\widehat{\varphi} + \bar{b}(\xi)\varphi, \\ \widehat{\psi} = \bar{a}(\xi)\varphi + b(\xi)\widehat{\varphi}, \end{cases} \tag{1.4}$$

where $a(\xi) = W\{\varphi, \psi\}$, $b(\xi) = W\{\widehat{\psi}, \varphi\}$. The following equalities are true

$$|a(\xi)|^2 + |b(\xi)|^2 = 1, \quad \bar{a}(\xi) = a(-\xi), \quad \bar{b}(\xi) = b(-\xi).$$

The coefficients $a(\xi)$ and $b(\xi)$ are continuous for $\operatorname{Im}\xi = 0$ and satisfy the asymptotic equalities:

$$a(\xi) = 1 + O(\xi^{-1}), \quad b(\xi) = O(\xi^{-1}), \quad |\xi| \to \infty.$$

The function $\psi(x,\xi)$ can be represented as follows (see [27, p. 33])

$$\psi(x,\xi) = \begin{pmatrix}0\\1\end{pmatrix} e^{i\xi x} + \int_x^\infty \mathbf{K}(x,s)e^{i\xi s}\, ds, \tag{1.5}$$

where $\mathbf{K}(x,s) = \begin{pmatrix}K_1(x,s)\\K_2(x,s)\end{pmatrix}$. In representation (1.5), the kernel $\mathbf{K}(x,s)$ does not depend on ξ and the equality

$$u(x) = -2K_1(x,x) \tag{1.6}$$

holds. The function $a(\xi)$ ($\bar{a}(\xi)$) continues analytically to the upper (lower) half-plane and has a finite number of zeros ξ_k ($\bar{\xi}_k$) there, where ξ_k ($\bar{\xi}_k$) is an eigenvalue of the operator $L(0)$, so that the following equality is true

$$\varphi(x,\xi_k) = C_k\psi(x,\xi_k) \quad \big(\widehat{\varphi}(x,\bar{\xi}_k) = \bar{C}_k\widehat{\psi}(x,\bar{\xi}_k)\big), \quad k = 1,2,\dots,N.$$

D e f i n i t i o n 1.1. The set of values $\left\{r^+(\xi) \equiv \frac{b(\xi)}{a(\xi)},\ \xi_k,\ C_k,\ k = 1,2,\dots,N\right\}$ is called *the scattering data for the operator* $L(0)$.

The components of the kernel $\mathbf{K}(x,y)$ in representation (1.5) for $y > x$ are solutions to the Gelfand–Levitan–Marchenko system of integral equations

$$K_2(x,y) + \int_x^\infty K_1(x,s)F(s+y)\, ds = 0,$$
$$-K_1(x,y) + F(x+y) + \int_x^\infty K_2(x,s)F(s+y)\, ds = 0,$$

where

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} r^+(\xi)e^{i\xi x}\,d\xi - i\sum_{j=1}^{N} C_j e^{i\xi_j x}.$$

Note that the following vector functions

$$h_n(x) = \frac{\frac{d}{d\xi}\left(\varphi - C_n\psi\right)\big|_{\xi=\xi_n}}{\dot{a}(\xi_n)}, \quad n = \overline{1,N}, \tag{1.7}$$

are solutions to the equations $L(0)h_n = \xi_n h_n$. Therefore, the following formula is valid:

$$h_n(x) = \frac{\beta_n}{\dot{a}(\xi_n)}\varphi(x,\xi_n) + \vartheta_n g_n, \quad n = \overline{1,N}.$$

In addition, the functions $h_n(x)$ have the following asymptotics,

$$h_n \sim -C_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi_n x}, \quad x \to -\infty; \qquad h_n \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi_n x}, \quad x \to \infty. \tag{1.8}$$

According to (1.8), we get the following equality

$$W\{\varphi_n, h_n\} \equiv \varphi_{n1}h_{n2} - \varphi_{n2}h_{n1} = -C_n, \quad n = \overline{1,N}.$$

The following lemmas are true.

L e m m a 1.1. *If* $Y(x,\xi) = \begin{pmatrix} y_1(x,\xi) \\ y_2(x,\xi) \end{pmatrix}$ *is a solution of equation* (1.2), *then* $\widehat{Y}(x,\xi) = \begin{pmatrix} y_2(x,-\xi) \\ -y_1(x,-\xi) \end{pmatrix}$ *satisfies the equation* $L\nu = -\xi\nu$.

L e m m a 1.2. *If vector functions* $Y = \begin{pmatrix} y_1(x,\xi) \\ y_2(x,\xi) \end{pmatrix}$ *and* $Z = \begin{pmatrix} z_1(x,\eta) \\ z_2(x,\eta) \end{pmatrix}$ *are solutions of equations* $LY = \xi Y$ *and* $LZ = \eta Z$, *then their components satisfy the equalities*

$$\frac{d}{dx}(y_1z_1 + y_2z_2) = -i(\xi+\eta)(y_1z_1 - y_2z_2),$$
$$\frac{d}{dx}(y_1z_2 - y_2z_1) = -i(\xi-\eta)(y_1z_2 + y_2z_1).$$

The validity of both lemmas is proved by a direct verification.
The following theorem is true.

T h e o r e m 1.1 (see [28, § 6.2, p. 353]). *The scattering data of the operator* L *uniquely determine* L.

§ 2. Evolution of scattering data

Let potential $u(x,t)$ in the system of equations $LY = \xi Y$ be a solution to the equation

$$u_t + p(t)(u_{xxx} + 6u^2u_x) = G(x,t), \tag{2.1}$$

where $G(x,t) = -q(t)u_x(x,t) + \sum\limits_{k=1}^{2N}\alpha_k(t)\big(f_{k1}g_{k1} - f_{k2}g_{k2}\big)$. The operator

$$A = p(t)\begin{pmatrix} -4i\xi^3 + 2iu^2\xi & 4u\xi^2 + 2iu_x\xi - 2u^3 - u_{xx} \\ -4u\xi^2 + 2iu_x\xi + 2u^3 + u_{xx} & 4i\xi^3 - 2iu^2\xi \end{pmatrix} \tag{2.2}$$

satisfies the following Lax relation

$$[L, A] \equiv LA - AL = i \begin{pmatrix} 0 & p(t)(-6u^2 u_x - u_{xxx}) \\ p(t)(-6u^2 u_x - u_{xxx}) & 0 \end{pmatrix}. \tag{2.3}$$

Therefore, equation (2.1) can be rewritten as

$$L_t + [L, A] = iR, \tag{2.4}$$

where $R = \begin{pmatrix} 0 & -G \\ -G & 0 \end{pmatrix}$. Differentiating the equality

$$L\varphi = \xi\varphi$$

with respect to t, we obtain

$$L_t\varphi + L\varphi_t = \xi\varphi_t,$$

which, according to (2.4), can be rewritten in the form

$$(L - \xi)(\varphi_t - A\varphi) = -iR\varphi.$$

Using the method of variation of constants, we can write

$$\varphi_t - A\varphi = B(x)\psi + D(x)\varphi. \tag{2.5}$$

Then, to determine $B(x)$ and $D(x)$, we obtain

$$MB_x\psi + MD_x\varphi = -R\varphi, \tag{2.6}$$

where $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. To solve equation (2.6), it is convenient to introduce the following notation $\widetilde{\varphi} = \begin{pmatrix} \varphi_2 \\ \varphi_1 \end{pmatrix}$, $\widetilde{\psi} = \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix}$. According to (2.3) and the definition of the Wronskian, the following equalities are valid:

$$\widetilde{\psi}^T M\varphi = -\widetilde{\varphi}^T M\psi = a, \qquad \widetilde{\psi}^T M\psi = \widetilde{\varphi}^T M\varphi = 0.$$

Multiplying (2.6) by $\widetilde{\varphi}^T$ and $\widetilde{\psi}^T$ we obtain

$$B_x = \frac{\widetilde{\varphi}^T R\varphi}{a}, \qquad D_x = -\frac{\widetilde{\psi}^T R\varphi}{a}. \tag{2.7}$$

According to (2.2), as $x \to -\infty$, we have

$$\varphi_t - A\varphi \to \begin{pmatrix} 4i\xi^3 p(t) & 0 \\ 0 & -4i\xi^3 p(t) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x} = \begin{pmatrix} 4i\xi^3 p(t) \\ 0 \end{pmatrix} e^{-i\xi x},$$

therefore, based on (2.5), we obtain

$$D(x) \to 4i\xi^3 p(t), \quad B(x) \to 0, \quad x \to -\infty.$$

Hence, from (2.7) we can determine

$$D(x) = -\frac{1}{a}\int_{-\infty}^{x} \widetilde{\psi}^T R\varphi \, dx + 4i\xi^3 p(t), \quad B(x) = \frac{1}{a}\int_{-\infty}^{x} \widetilde{\varphi}^T R\varphi \, dx.$$

Thus, equality (2.5) has the following form:

$$\varphi_t - A\varphi = \Big(\frac{1}{a}\int_{-\infty}^{x} \widetilde{\varphi}^T R\varphi\,dx\Big)\psi + \Big(-\frac{1}{a}\int_{-\infty}^{x} \widetilde{\psi}^T R\varphi\,dx + 4i\xi^3 p(t)\Big)\varphi. \tag{2.8}$$

According to (1.4), equality (2.8) can be rewritten in the following form

$$a_t\widehat{\psi} + b_t\psi - A\Big(a\widehat{\psi} + b\psi\Big) = \Big(\frac{1}{a}\int_{-\infty}^{x} \widetilde{\varphi}^T R\varphi\,dx\Big)\psi + \Big(-\frac{1}{a}\int_{-\infty}^{x} \widetilde{\psi}^T R\varphi\,dx + 4i\xi^3 p(t)\Big)\Big(a\widehat{\psi} + b\psi\Big).$$

Passing in the last equality to the limit as $x \to +\infty$ and taking into account (2.2), we obtain

$$a_t = -\int_{-\infty}^{\infty} \widetilde{\psi}^T R\varphi\,dx,$$

$$b_t = \frac{1}{a}\int_{-\infty}^{\infty} \widetilde{\varphi}^T R\varphi\,dx - \frac{b}{a}\int_{-\infty}^{\infty} \widetilde{\psi}^T R\varphi\,dx + 8i\xi^3 p(t)b.$$

Therefore, at $\operatorname{Im}\xi = 0$ we have

$$\frac{dr^+}{dt} = 8i\xi^3 p(t)r^+ - \frac{1}{a^2}\int_{-\infty}^{\infty} G\big(\varphi_1^2 + \varphi_2^2\big)\,dx. \tag{2.9}$$

L e m m a 2.1. *If vector-function* $\varphi = \begin{pmatrix}\varphi_1(x,\xi)\\ \varphi_2(x,\xi)\end{pmatrix}$ *is a solution to the system of equations* (1.1)*, then its components satisfy the following equality*

$$\int_{-\infty}^{\infty} G(\varphi_1^2 + \varphi_2^2)\,dx = -2i\xi a(\xi)b(\xi)\sum_{k=1}^{N}\frac{\alpha_k(t)\omega_k(t)}{\xi^2 - \xi_k^2} + 2i\xi q(t)a(\xi)b(\xi). \tag{2.10}$$

P r o o f. Let the potential $u(x,t)$ in the system of equations (0.1) be a solution to the equation

$$u_t + p(t)(6u^2u_x + u_{xxx}) = G(x,t),$$

where G rather quickly tends to zero at $x \to \pm\infty$.

According to Lemma 1.1 and the first of the conditions (0.4), the right-hand side in the equation (0.1) can be rewritten in the form

$$\sum_{k=1}^{2N}\alpha_k(t)\big(f_{k1}g_{k1} - f_{k2}g_{k2}\big) = 2\sum_{\substack{k=1,\\ \operatorname{Im}\xi_k>0}}^{N}\alpha_k(t)\big(f_{k1}g_{k1} - f_{k2}g_{k2}\big).$$

According to Lemma 1.2, we have the following equality

$$\begin{aligned}
&\alpha_k(t)(f_{k1}g_{k1} - f_{k2}g_{k2})(\varphi_1^2 + \varphi_2^2) = \\
&= \alpha_k(t)f_{k1}g_{k1}\varphi_1^2 + \alpha_k(t)f_{k1}g_{k1}\varphi_2^2 - \alpha_k(t)f_{k2}g_{k2}\varphi_1^2 - \alpha_k(t)f_{k2}g_{k2}\varphi_2^2 = \\
&= \frac{\alpha_k(t)}{2}\big[(f_{k1}\varphi_1 - f_{k2}\varphi_2)(g_{k1}\varphi_1 + g_{k2}\varphi_2) + (f_{k1}\varphi_1 + f_{k2}\varphi_2)(g_{k1}\varphi_1 - g_{k2}\varphi_2)\big] + \\
&+ \frac{\alpha_k(t)}{2}\big[(f_{k1}\varphi_2 - f_{k2}\varphi_1)(g_{k2}\varphi_1 + g_{k1}\varphi_2) - (f_{k1}\varphi_2 + f_{k2}\varphi_1)(g_{k2}\varphi_1 - g_{k1}\varphi_2)\big] = \\
&= \frac{\alpha_k(t)}{-2i(\xi + \xi_k)}\frac{d}{dx}\big[(f_{k1}\varphi_1 + f_{k2}\varphi_2)(g_{k1}\varphi_1 + g_{k2}\varphi_2)\big] + \\
&+ \frac{\alpha_k(t)}{2i(\xi - \xi_k)}\frac{d}{dx}\big[(\varphi_1 f_{k2} - \varphi_2 f_{k1})(\varphi_1 g_{k2} - \varphi_2 g_{k1})\big].
\end{aligned}$$

The following asymptotics are valid: for $x \to -\infty$,

$$\begin{aligned}
\varphi &\sim \begin{pmatrix}1\\0\end{pmatrix} e^{-i\xi x}, \qquad \psi \sim \begin{pmatrix}\overline{b}e^{-i\xi x}\\ ae^{i\xi x}\end{pmatrix},\\
g_k &\sim \begin{pmatrix}0\\1\end{pmatrix} \omega_k(t)e^{i\xi_k x}, \qquad \psi_k \sim \frac{1}{C_k}\begin{pmatrix}1\\0\end{pmatrix} e^{-i\xi_k x},
\end{aligned} \tag{2.11}$$

and, for $x \to +\infty$,

$$\begin{aligned}
\varphi &\sim \begin{pmatrix}ae^{-i\xi x}\\ be^{i\xi x}\end{pmatrix}, \qquad \psi \sim \begin{pmatrix}0\\1\end{pmatrix} e^{i\xi x},\\
g_k &\sim -\begin{pmatrix}1\\0\end{pmatrix} \frac{\omega_k(t)}{C_k}e^{-i\xi_k x}, \qquad \varphi_k \sim C_k\begin{pmatrix}0\\1\end{pmatrix} e^{i\xi_k x}.
\end{aligned} \tag{2.12}$$

Integrating the last equation from $-\infty$ to $+\infty$, and then using the asymptotics (2.11), (2.12), we obtain the following equalities:

$$\frac{\alpha_k(t)}{-2i(\xi+\xi_k)}\int_{-\infty}^{\infty}\frac{d}{dx}\big[(f_{k1}\varphi_1+f_{k2}\varphi_2)(g_{k1}\varphi_1+g_{k2}\varphi_2)\big]\,dx =$$

$$=\frac{\alpha_k(t)}{-2i(\xi+\xi_k)}\lim_{R\to\infty} C_k b(\xi)e^{i(\xi+\xi_k)R}\Big(-\frac{\omega_k(t)}{C_k}a(\xi)e^{-i(\xi+\xi_k)R}\Big) = \frac{a(\xi)b(\xi)\omega_k(t)\alpha_k(t)}{2i(\xi+\xi_k)},$$

$$\frac{\alpha_k(t)}{2i(\xi-\xi_k)}\int_{-\infty}^{\infty}\frac{d}{dx}\big[(\varphi_1 f_{k2}-\varphi_2 f_{k1})(\varphi_1 g_{k2}-\varphi_2 g_{k1})\big]\,dx =$$

$$=\frac{\alpha_k(t)}{2i(\xi-\xi_k)}\lim_{R\to\infty}\big(a(\xi)C_k e^{-i(\xi-\xi_k)R}\big)\Big(\frac{b(\xi)\omega_k(t)}{C_k}e^{i(\xi-\xi_k)R}\Big) = \frac{a(\xi)b(\xi)\omega_k(t)\alpha_k(t)}{2i(\xi-\xi_k)}.$$

The following integral is calculated in the same way:

$$-\int_{-\infty}^{\infty} q(t)u_x\big(\varphi_1^2+\varphi_2^2\big)\,dx = -q(t)\int_{-\infty}^{\infty}\big(\varphi_1^2+\varphi_2^2\big)\,du =$$

$$= -q(t)\;u\big(\varphi_1^2+\varphi_2^2\big)\big|_{-\infty}^{\infty} + q(t)\int_{-\infty}^{\infty} u\big(\varphi_1^2+\varphi_2^2\big)'\,dx = 2q(t)\int_{-\infty}^{\infty}\big(u\varphi_1\varphi_1'+u\varphi_2\varphi_2'\big)\,dx =$$

$$= 2q(t)\int_{-\infty}^{\infty}\big[(-\varphi_2'+i\xi\varphi_2)\varphi_1'+(\varphi_1'+i\xi\varphi_1)\varphi_2'\big]\,dx =$$

$$= 2q(t)\int_{-\infty}^{\infty}\big[-\varphi_1'\varphi_2'+i\xi\varphi_1'\varphi_2+\varphi_1'\varphi_2'+i\xi\varphi_1\varphi_2'\big]\,dx = 2i\xi q(t)\int_{-\infty}^{\infty}(\varphi_1\varphi_2\,dx =$$

$$= 2i\xi q(t)\lim_{R\to\infty}(\varphi_1\varphi_2)\Big|_{-R}^{R} = 2i\xi q(t)a(\xi)b(\xi).$$

Consequently, we obtain the equality (2.10)

$$\int_{-\infty}^{\infty} G(\varphi_1^2+\varphi_2^2)\,dx = -2i\xi a(\xi)b(\xi)\sum_{k=1}^{N}\frac{\alpha_k(t)\omega_k(t)}{\xi^2-\xi_k^2} + 2i\xi q(t)a(\xi)b(\xi). \qquad \square$$

According to equalities (2.9) and (2.10), we have the following equation:

$$\frac{dr^+}{dt} = \left[8i\xi^3 p(t) + 2i\xi\sum_{k=1}^{N}\frac{\alpha_k(t)\omega_k(t)}{\xi^2-\xi_k^2} - 2i\xi q(t)\right] r^+ \quad (\operatorname{Im}\xi = 0).$$

Differentiating the equality $\varphi_n = C_n\psi_n$ with respect to t, we get the following relation

$$\frac{\partial\varphi}{\partial t}\bigg|_{\xi=\xi_n} + \frac{\partial\varphi}{\partial\xi}\bigg|_{\xi=\xi_n}\frac{d\xi_n}{dt} = \frac{dC_n}{dt}\psi_n + C_n\frac{\partial\psi}{\partial t}\bigg|_{\xi=\xi_n} + C_n\frac{\partial\psi}{\partial\xi}\bigg|_{\xi=\xi_n}\frac{d\xi_n}{dt},$$

which, according to (1.7), can be rewritten in the form

$$\frac{\partial \varphi_n}{\partial t} = \frac{dC_n}{dt}\psi_n + C_n\frac{\partial \psi_n}{\partial t} - \dot{a}(\xi_n)h_n\frac{d\xi_n}{dt}, \tag{2.13}$$

where $\frac{\partial \varphi_n}{\partial t} = \left.\frac{\partial \varphi}{\partial t}\right|_{\xi=\xi_n}$.

Similarly to the case of a continuous spectrum, taking into account (2.7), in the case of a discrete spectrum, we obtain the following equality:

$$\frac{\partial \varphi_n}{\partial t} - A\varphi_n = \left(-\frac{1}{C_n}\int_{-\infty}^{x} \widetilde{\varphi}_n^T R\varphi_n\,dx\right)h_n + \left(\frac{1}{C_n}\int_{-\infty}^{x} \widetilde{h}_n^T R\varphi_n\,dx + 4i\xi_n^3 p(t)\right)\varphi_n, \tag{2.14}$$

where $\widetilde{h_n} = \begin{pmatrix} h_{n_2} \\ h_{n_1}\end{pmatrix}$. According to (2.13), the last equality can be rewritten in the following form:

$$\begin{gathered}\frac{\partial C_n}{\partial t}\psi_n + C_n\frac{\partial \psi_n}{\partial t} - \dot{a}(\xi_n)\frac{d\xi_n}{dt}h_n - C_nA\psi_n = \\ = \left(-\frac{1}{C_n}\int_{-\infty}^{x} \widetilde{\varphi}_n^T R\varphi_n\,dx\right)h_n + \left(\frac{1}{C_n}\int_{-\infty}^{x} \widetilde{h}_n^T R\varphi_n\,dx + 4i\xi_n^3 p(t)\right)C_n\psi_n.\end{gathered}$$

Passing in this equality to the limit as $x \to +\infty$, taking into account (1.8) and (2.2), we obtain the following equalities:

$$\frac{dC_n}{dt} = \left(8i\xi_n^3 p(t) + \int_{-\infty}^{\infty} \widetilde{h}_n^T R\psi_n\,dx\right)C_n, \qquad \frac{d\xi_n}{dt} = \frac{\int_{-\infty}^{\infty} \widetilde{\varphi}_n^T R\varphi_n\,dx}{C_n\dot{a}(\xi_n)}.$$

Thus, we have the following identities

$$\begin{gathered}\frac{dC_n}{dt} = \left(8i\xi_n^3 p(t) - \int_{-\infty}^{\infty} G(h_{n1}\psi_{n1} + h_{n2}\psi_{n2})\,dx\right)C_n, \\ \frac{d\xi_n}{dt} = \frac{-\int_{-\infty}^{\infty} G(\varphi_{n1}^2 + \varphi_{n2}^2)\,dx}{C_n\dot{a}(\xi_n)}.\end{gathered} \tag{2.15}$$

It remains to note that, according to the identity

$$\dot{a}(\xi_n) = -\frac{2i}{C_n}\int_{-\infty}^{+\infty} \varphi_{n1}\varphi_{n2}\,dx,$$

the last equality can be rewritten as

$$\frac{d\xi_n}{dt} = \frac{\int_{-\infty}^{\infty} G(\varphi_{n1}^2 + \varphi_{n2}^2)\,dx}{2i\int_{-\infty}^{+\infty} \varphi_{n1}\varphi_{n2}\,dx}. \tag{2.16}$$

L e m m a 2.2. *If the vector-functions* $\varphi_n(x,\xi_n) = \begin{pmatrix}\varphi_{n_1}(x,\xi_n) \\ \varphi_{n_2}(x,\xi_n)\end{pmatrix}$, $\psi_n(x,\xi_n) = \begin{pmatrix}\psi_{n_1}(x,\xi_n) \\ \psi_{n_2}(x,\xi_n)\end{pmatrix}$ *and* $h_n(x,\xi_n) = \begin{pmatrix}h_{n_1}(x,\xi_n) \\ h_{n_2}(x,\xi_n)\end{pmatrix}$ *are the solutions of the equation* $L\nu = \xi_n\nu$, *then their components satisfy the equalities*

$$\int_{-\infty}^{\infty} G(h_{n_1}\psi_{n_1} + h_{n_2}\psi_{n_2})\,dx = i\alpha_n(t)\beta_n(t)\omega_n(t) + 2i\xi_n q(t), \tag{2.17}$$

$$\int_{-\infty}^{\infty} G(\varphi_{n1}^2 + \varphi_{n2}^2)\,dx = -2\omega_n(t)\alpha_n(t)\int_{-\infty}^{\infty} \varphi_{n1}\varphi_{n2}\,dx. \tag{2.18}$$

P r o o f. Firstly, to prove the lemma, we write the following equality:

$$\int_{-\infty}^{+\infty} G(h_{n_1}\psi_{n_1} + h_{n_2}\psi_{n_2})\,dx = -q(t)\int_{-\infty}^{+\infty} u_x(x,t)(h_{n_1}\psi_{n_1} + h_{n_2}\psi_{n_2})\,dx + \\ + 2\sum_{k=1}^{N}\alpha_k(t)\int_{-\infty}^{+\infty}(f_{k1}g_{k1} - f_{k2}g_{k2})(h_{n_1}\psi_{n_1} + h_{n_2}\psi_{n_2})\,dx. \tag{2.19}$$

At $\xi_k \neq \xi_n$, according to Lemma 1.2, we have

$$\alpha_k(f_{k1}g_{k1} - f_{k2}g_{k2})(h_{n1}\psi_{n1} + h_{n2}\psi_{n2}) = \alpha_k f_{k1}g_{k1}h_{n1}\psi_{n1} + \alpha_k f_{k1}g_{k1}h_{n2}\psi_{n2} - \\ - \alpha_k f_{k2}g_{k2}h_{n1}\psi_{n1} - \alpha_k f_{k2}g_{k2}h_{n2}\psi_{n2} = \frac{\alpha_k}{-2i(\xi_n+\xi_k)}\frac{d}{dx}\big[(h_{n1}f_{k1} + h_{n2}f_{k2})(\psi_{n1}g_{k1} + \psi_{n2}g_{k1})\big] + \\ + \frac{\alpha_k}{2i(\xi_n-\xi_k)}\frac{d}{dx}\big[(h_{n1}f_{k2} - h_{n2}f_{k1})(\psi_{n1}g_{k2} - \psi_{k2}g_{k1})\big].$$

Let's integrate the above equality over x from $-\infty$ to $+\infty$:

$$\frac{\alpha_k(t)}{-2i(\xi_n+\xi_k)}\int_{-\infty}^{\infty}\frac{d}{dx}\big[(h_{n1}f_{k1} + h_{n2}f_{k2})(\psi_{n1}g_{k1} + \psi_{n2}g_{k2})\big]\,dx + \\ + \frac{\alpha_k(t)}{2i(\xi_n-\xi_k)}\int_{-\infty}^{\infty}\frac{d}{dx}\big[(h_{n1}f_{k2} - h_{n2}f_{k1})(\psi_{n1}g_{k2} - \psi_{n2}g_{k1})\big]\,dx = \\ = \frac{\alpha_k(t)}{-2i(\xi_n+\xi_k)}\lim_{R\to\infty}\big[(h_{n1}f_{k1} + h_{n2}f_{k2})(\psi_{n1}g_{k1} + \psi_{n2}g_{k2})\big]\Big|_{-R}^{R} + \\ + \frac{\alpha_k(t)}{2i(\xi_n-\xi_k)}\lim_{R\to\infty}\big[(h_{n1}f_{k2} - h_{n2}f_{k1})(\psi_{n1}g_{k2} + \psi_{n2}g_{k1})\big]\Big|_{-R}^{R} = \\ = \frac{\alpha_k(t)}{2i(\xi_n-\xi_k)}\lim_{R\to\infty}\Big[C_k e^{i(-\xi_n+\xi_k)R}\frac{\omega_k(t)}{C_k}e^{i(\xi_n-\xi_k)R} - C_n e^{i(\xi_n-\xi_k)R}\frac{\omega_k(t)}{C_n}e^{-i(\xi_n-\xi_k)R}\Big] = 0.$$

Therefore, for $\xi_k \neq \xi_n$ we have

$$\alpha_k(t)\int_{-\infty}^{\infty}(f_{k1}g_{k1} - f_{k2}g_{k2})(h_{n1}\psi_{n1} + h_{n2}\psi_{n2})\,dx = 0.$$

If $\xi_k = \xi_n$, then

$$\alpha_n(t)(f_{n1}g_{n1} - f_{n2}g_{n2})(h_{n1}\psi_{n1} + h_{n2}\psi_{n2}) = -\frac{\alpha_n(t)}{4i\xi_n}\frac{d}{dx}\big[(h_{n1}g_{n1} + h_{n2}g_{n2})(f_{n1}\psi_{n1} + f_{n2}\psi_{n2})\big] - \\ - \frac{\alpha_n(t)}{2}\big[(\psi_{n1}f_{n2} - \psi_{n2}f_{n1})(h_{n1}g_{n2} + h_{n2}g_{n1}) + (\psi_{n1}f_{n2} + \psi_{n2}f_{n1})(h_{n1}g_{n2} - h_{n2}g_{n1})\big] = \\ = -C_n\psi_{n1}\psi_{n2}\alpha_n(t)\bigg[\Big(\frac{\beta_n(t)}{\dot a(\xi_n)}\varphi_{n1} + \vartheta_n(t)g_{n1}\Big)g_{n2} - \Big(\frac{\beta_n(t)}{\dot a(\xi_n)}\varphi_{n2} + \vartheta_n(t)g_{n2}\Big)g_{n1}\bigg] = \\ = -C_n\psi_{n1}\psi_{n2}\frac{\beta_n(t)}{\dot a(\xi_n)}\alpha_n(t)\omega_n(t). \tag{2.20}$$

Let us integrate equality (2.20) with respect to x:

$$\alpha_n(t)\int_{-\infty}^{\infty}(f_{n1}g_{n1} - f_{n2}g_{n2})(h_{n1}\psi_{n1} + h_{n2}\psi_{n2})\,dx = -\frac{\beta_n(t)}{\dot a(\xi_n)}\alpha_n(t)\omega_n(t)\int_{-\infty}^{\infty}C_n\psi_{n1}\psi_{n2}\,dx = \\ = -\frac{\beta_n(t)}{\dot a(\xi_n)}\frac{\alpha_n(t)\omega_n(t)}{C_n}\int_{-\infty}^{\infty}\varphi_{n1}\varphi_{n2}\,dx = -\frac{i}{2}\alpha_n(t)\beta_n(t)\omega_n(t). \tag{2.21}$$

Let us calculate the following integral using equality (1.1) and asymptotics (1.3), (1.8):

$$-q(t)\int_{-\infty}^{\infty} u_x(h_{n_1}\psi_{n_1}+h_{n_2}\psi_{n_2})\,dx = -q(t)\int_{-\infty}^{\infty}(h_{n_1}\psi_{n_1}+h_{n_2}\psi_{n_2})\,du =$$
$$= -q(t)\;u(h_{n_1}\psi_{n_1}+h_{n_2}\psi_{n_2}|_{-\infty}^{\infty} + q(t)\int_{-\infty}^{\infty}(uh'_{n_1}\psi_{n_1}+uh_{n_1}\psi'_{n_1})\,dx +$$
$$+\,q(t)\int_{-\infty}^{\infty}(uh'_{n_2}\psi_{n_2}+uh_{n_2}\psi'_{n_2})\,dx =$$
$$= q(t)\int_{-\infty}^{\infty}\Big((-\psi'_{n_2}+i\xi_n\psi_{n_2})h'_{n_1}+(-h'_{n_2}+i\xi_n h_{n_2})\psi'_{n_1}\Big)\,dx +$$
$$+\,q(t)\int_{-\infty}^{\infty}\Big(h'_{n_2}(\psi'_{n_1}+i\xi_n\psi_{n_1})+\psi'_{n_2}(h'_{n_1}+i\xi_n h_{n_1})\Big)\,dx =$$
$$= i\xi_n q(t)\int_{-\infty}^{\infty}\Big((h_{n_1}\psi_{n_2})'+(h_{n_2}\psi_{n_1})'\Big)\,dx = i\xi_n q(t)\;(h_{n_1}\psi_{n_2}+h_{n_2}\psi_{n_1})|_{-\infty}^{\infty} =$$
$$= i\xi_n q(t)\Big(e^{-i\xi_n x}\cdot e^{i\xi_n x} - \Big(-C_n e^{i\xi_n x}\cdot\frac{1}{C_n}e^{-i\xi_n x}\Big)\Big) = 2i\xi_n q(t).$$

Using the last equality and equalities (2.21), we obtain identity (2.17). Now, we derive the equality (2.18). At $\xi_k \neq \xi_n$, according to Lemma 1.2, we have

$$\alpha_k(t)\int_{-\infty}^{\infty}(f_{k1}g_{k1}-f_{k2}g_{k2})(\varphi_{n1}^2+\varphi_{n2}^2)\,dx =$$
$$= \frac{\alpha_k(t)}{-2i(\xi_n+\xi_k)}\int_{-\infty}^{\infty}\frac{d}{dx}\big[(f_{k1}\varphi_{n1}+\varphi_{n2}f_{k2})(\varphi_{n1}g_{k1}+\varphi_{n2}g_{n2})\big]\,dx +$$
$$+\,\frac{\alpha_k(t)}{2i(\xi_n-\xi_k)}\int_{-\infty}^{\infty}\frac{d}{dx}\big[(f_{k2}\varphi_{n1}-\varphi_{n2}f_{k1})(\varphi_{n1}g_{k2}-\varphi_{n2}g_{n1})\big]\,dx =$$
$$= \frac{-\alpha_k(t)}{2i(\xi_n+\xi_k)}\lim_{R\to\infty}\Big[(f_{k1}\varphi_{n1}+\varphi_{n2}f_{k2})(\varphi_{n1}g_{k1}+\varphi_{n2}g_{n2})\Big|_{-R}^{R}\Big] +$$
$$+\,\frac{\alpha_k(t)}{2i(\xi_n-\xi_k)}\lim_{R\to\infty}\Big[(f_{k2}\varphi_{n1}-\varphi_{n2}f_{k1})(\varphi_{n1}g_{k2}-\varphi_{n2}g_{k1})\Big|_{-R}^{R}\Big] = 0.$$

If $\xi_k = \xi_n$, then

$$\alpha_n(t)\int_{-\infty}^{\infty}(f_{n1}g_{n1}-f_{n2}g_{n2})(\varphi_{n1}^2+\varphi_{n2}^2)\,dx =$$
$$= \frac{\alpha_n(t)}{-4i\xi_n}\int_{-\infty}^{\infty}\frac{d}{dx}\big[(f_{n1}\varphi_{n1}+\varphi_{n2}f_{n2})(\varphi_{n1}g_{n1}+\varphi_{n2}g_{n2})\big]\,dx -$$
$$-\,\frac{\alpha_n(t)}{2}\int_{-\infty}^{\infty}\big[(f_{n1}\varphi_{n2}-\varphi_{n1}f_{n2})(\varphi_{n1}g_{n2}+\varphi_{n2}g_{n1})\big]\,dx -$$
$$-\,\frac{\alpha_n(t)}{2}\int_{-\infty}^{\infty}\big[(f_{n2}\varphi_{n1}+\varphi_{n2}f_{n1})(\varphi_{n1}g_{n2}-\varphi_{n2}g_{n1})\big]\,dx =$$
$$= -\frac{\alpha_n(t)}{2}\int_{-\infty}^{\infty}2\varphi_{n1}\varphi_{n2}(f_{n1}g_{n2}-f_{n2}g_{n1})\,dx = -\alpha_n(t)\omega_n(t)\int_{-\infty}^{\infty}\varphi_{n1}\varphi_{n2}\,dx.$$

So, we got this equality

$$\alpha_n(t)\int_{-\infty}^{\infty}(f_{n1}g_{n1}-f_{n2}g_{n2})(\varphi_{n1}^2+\varphi_{n2}^2)\,dx = -\alpha_n(t)\omega_n(t)\int_{-\infty}^{\infty}\varphi_{n1}\varphi_{n2}\,dx. \tag{2.22}$$

In a similar way, it can be shown that

$$\int_{-\infty}^{+\infty} q(t)u_x(\varphi_{n_1}^2 + \varphi_{n_2}^2)dx = 0. \tag{2.23}$$

Using equalities (2.22) and (2.23), we obtain

$$\int_{-\infty}^{\infty} G(\varphi_{n1}^2 + \varphi_{n2}^2)\, dx = -2\omega_n(t)\alpha_n(t) \int_{-\infty}^{\infty} \varphi_{n1}\varphi_{n2}\, dx. \qquad \square$$

Substituting equality (2.17) into the right side of equality (2.15), we obtain the following expression:

$$\frac{dC_n}{dt} = [8i\xi_n^3 p(t) + i\alpha_n(t)\beta_n(t)\omega_n(t) - 2i\xi_n q(t)]C_n, \qquad n = \overline{1, N}.$$

According to equalities (2.16) and (2.18), we can calculate the evolution of the eigenvalue

$$\frac{d\xi_n}{dt} = i\alpha_n(t)\omega_n(t), \qquad n = \overline{1, N}.$$

Thus, we have proved the following theorem.

T h e o r e m 2.1. *If the functions $u(x,t)$, $f_k(x,t)$, $g_k(x,t)$, $k = \overline{1, N}$, are a solution to the problem* (0.1)–(0.5)*, then the scattering data of the operator $L(t)$ with the potential $u(x,t)$ satisfy the following differential equations*

$$\frac{d\xi_n}{dt} = i\alpha_n(t)\omega_n(t), \qquad n = \overline{1, N},$$

$$\frac{dr^+}{dt} = \left[8\, i\xi^3 p(t) + 2i\xi \sum_{k=1}^{N} \frac{\alpha_k(t)\omega_k(t)}{\xi^2 - \xi_k^2} - 2i\xi q(t)\right] r^+ \qquad (\operatorname{Im} \xi = 0),$$

$$\frac{dC_n}{dt} = \left[8i\xi_n^3 p(t) + i\alpha_n(t)\beta_n(t)\omega_n(t) - 2i\xi_n q(t)\right] C_n, \qquad n = \overline{1, N}.$$

The obtained equalities completely determine the evolution of the scattering data, which makes it possible to apply the inverse scattering method to solve problem (0.1)–(0.5).

E x a m p l e 2.1. Consider the following Cauchy problem

$$u_t + \frac{27(6u^2u_x + u_{xxx})}{(t+1)(2t^3 + 3t^2 + 3)^3} + \frac{(3 - 3i(t^2 + t)^2)u_x}{(t+1)(2t^3 + 3t^2 + 3)} = 2(t+1)(f_{11}g_{11} - f_{12}g_{12}),$$

$$L(t)f_1 = \xi_1 f_1, \qquad L(t)g_1 = \xi_1 g_1,$$

$$u(x, 0) = -\frac{1}{\operatorname{ch} x}, \qquad \omega_1(t) = f_{11}g_{12} - f_{12}g_{11} = t.$$

It is easy to find the scattering data for the operator $L(0)$:

$$\left\{ r^+(0) = 0, \quad \xi_1(0) = \frac{i}{2}, \quad C_1(0) = i \right\}.$$

According to Theorem 2.1, the evolution of scattering data is as follows

$$\xi_1(t) = i\gamma(t), \quad r^+(t) = 0, \quad C_1(t) = ie^{\mu(t)},$$

where

$$\gamma(t) = \frac{t^3}{3} + \frac{t^2}{2} + \frac{1}{2}, \qquad \mu(t) = 2\ln(t+1).$$

Applying the inverse problem method, we obtain the following relations

$$f_{11}(x,t)=\frac{6(2t^3+3t^2+3)^2(t+1)^2e^{-(\frac{3}{2}+\frac{2t^3}{3}+t^2)x}}{(2t^3+3t^2+6)\big(4t^6+12t^5+9t^4+12t^3+18t^2+9+9(t+1)^4e^{-(2+2t^2+\frac{4}{3}t^3)x}\big)},$$

$$f_{12}(x,t)=e^{-\frac{x}{2}}-\frac{(36t^3+54t^2+54)(t+1)^2e^{-(\frac{5}{2}+\frac{4t^3}{3}+2t^2)x}}{(2t^3+3t^2+6)\big(4t^6+12t^5+9t^4+12t^3+18t^2+9+9(t+1)^4e^{-(2+2t^2+\frac{4}{3}t^3)x}\big)},$$

$$g_{11}(x,t)=te^{\frac{x}{2}+\int_0^x A(s,t)\,ds}\int_0^x\sigma(s,t)e^{-\frac{s}{2}-\int_0^s A(z,t)\,dz}\,ds,$$

$$g_{12}(x,t)=\frac{8\gamma^3(t)e^{2\gamma(t)x}-2\gamma(t)(t+1)^4e^{-2\gamma(t)x}+4\gamma^2(t)e^{2\gamma(t)x}+(t+1)^4e^{-2\gamma(t)x}}{8\gamma^2(t)(t+1)^2}\times$$
$$\times\, te^{\frac{x}{2}+\int_0^x A(s,t)\,ds}\int_0^x\sigma(s,t)e^{-\frac{s}{2}-\int_0^s A(z,t)\,dz}\,ds+$$
$$+\frac{t(2\gamma(t)+1)(4\gamma^2(t)+(t+1)^4e^{-4\gamma(t)x})}{8\gamma^2(t)(t+1)^2e^{-\left(2\gamma(t)+\frac{1}{2}\right)x}},$$

$$u(x,t)=\frac{-4\gamma^2(t)(t+1)^2}{(2\gamma^2(t)-0.5)e^{2\gamma(t)x}+(t+1)^2\operatorname{ch}(2\gamma(t)x-2\ln(t+1))}$$

where

$$A(x,t)=\frac{(6t^3+9t^2)(t+1)^2-\left(\frac{2}{3}t^3+t^2+2\right)(2t^3+3t^2+2)^2e^{(\frac{4}{3}t^3+2t^2+2)x}}{(2t^3+3t^2+2)^2e^{(\frac{4}{3}t^3+2t^2+2)x}+9(t+1)^2},$$

$$\sigma(x,t)=\frac{-e^{\frac{x}{2}}(2t^3+3t^2+6)\big((2t^3+3t^2+3)^2+9(t+1)^4e^{-(\frac{4}{3}t^3+2t^2+2)x}\big)}{3(2t^3+3t^2+3)^2+54(t+1)^4e^{-(\frac{4}{3}t^3+2t^2+2)x}}.$$

§ 3. Loaded mKdV equation with source

Consider the following equation:

$$u_t+P(u(x_0,t))(6u^2u_x+u_{xxx})+Q(u(x_1,t))u_x=\sum_{k=1}^{2N}B_k(u(x_2,t))(f_{k1}g_{k1}-f_{k2}g_{k2}),\qquad(3.1)$$

where $P(y)$, $Q(z)$ and $B_k(s)$, $k=\overline{1,2N}$, are polynomials in y, z and s, respectively. The equation (3.1) is not a particular case of the equation (0.1), because the coefficients in the equation (3.1) depend on the value of the solution on a manifold of lower dimension. Such equations are called loaded equations.

In the work [29], Nakhushev gave the most general definition of loaded equations and gave a detailed classification of various types of loaded equations. Among the works devoted to loaded equations, the works [30–38] should be singled out.

If in the problem (0.1)–(0.5) instead of the equation (0.1) we consider the equation (3.1), then the following theorem holds.

T h e o r e m 3.1. *If functions $u(x,t)$, $f_k(x,t)$, $g_k(x,t)$, $k=\overline{1,N}$, are a solution to the problem* (3.1), (0.2)–(0.5)*, in the class of functions* (0.5)*, then the scattering data of the operator $L(t)$ with potential $u(x,t)$ change according to t as follows*

$$\frac{d\xi_n}{dt}=iB_n(u(x_2,t))\omega_n(t),\qquad n=\overline{1,N},$$

$$\frac{dr^+}{dt}=\left[8\,i\xi^3P(u(x_0,t))+2i\xi\sum_{k=1}^{N}\frac{B_k(u(x_2,t))\omega_k(t)}{\xi^2-\xi_k^2}-2i\xi Q(u(x_1,t))\right]r^+\qquad(\operatorname{Im}\xi=0),$$

$$\frac{dC_n}{dt}=\big[8i\xi_n^3P(u(x_0,t))+iB_n(u(x_2,t))\beta_n(t)\omega_n(t)-2i\xi_nQ(u(x_1,t))\big]C_n,\qquad n=\overline{1,N}.$$

E x a m p l e 3.1. Consider the following Cauchy problem

$$u_t + 6u^2 u_x + u_{xxx} + \alpha(t)u(1,t)u_x = 2\rho(t)u(0,t)(f_{11}g_{11} - f_{12}g_{12}),$$
$$Lf_1 = \xi_1 f_1, \quad Lg_1 = \xi_1 g_1,$$
$$u(x,0) = -\frac{1}{\operatorname{ch} x}, \quad \omega_1(t) = f_{11}g_{12} - f_{12}g_{11} = t,$$

where

$$\rho(t) = \frac{(t+1)^2\left((3t^4+8t^3+6t^2+6)^2 e^{-10t} + 36e^{10t}\right)}{-2(3t^4+8t^3+6t^2+6)^2},$$
$$\alpha(t) = \frac{(10 - 8\gamma^3(t) + i(t^2+t)^2)\big(4\gamma^2(t)e^{-10t+2\gamma(t)} + e^{10t-2\gamma(t)}\big)}{-16\gamma^3(t)},$$
$$\gamma(t) = \frac{t^4}{4} + \frac{2t^3}{3} + \frac{t^2}{2} + \frac{1}{2}.$$

As in Example 2.1, the scattering data of the operator $L(0)$ have the form:

$$\Big\{r^+(0) = 0, \quad \xi_1(0) = \frac{i}{2}, \quad C_1(0) = i\Big\}.$$

According to Theorem 3.1, we have

$$\xi_1(t) = i\gamma(t), \quad r^+(t) = 0, \quad C_1(t) = ie^{\mu(t)},$$

where

$$\mu(t) = 8\int_0^t \gamma^3(\tau)\,d\tau + i\int_0^t \tau\rho(\tau)u(0,\tau)\omega_1(\tau)\,d\tau + 2\int_0^t \gamma(\tau)\alpha(\tau)u(1,\tau)\,d\tau. \tag{3.2}$$

Consequently, $F(x,t) = e^{-\gamma(t)x+\mu(t)}$. Solving the system of integral equations of Gelfand–Levitan–Marchenko, one can obtain

$$K_1(x,y) = \frac{4\gamma^2(t)e^{\mu(t)-\gamma(t)(x+y)}}{4\gamma^2(t) + e^{2\mu(t)-4\gamma(t)x}}.$$

Using the last equality and formula (1.6), we obtain the following:

$$u(x,t) = \frac{-4\gamma^2(t)}{(2\gamma^2(t)-\frac{1}{2})e^{-\mu(t)+2\gamma(t)x} + \operatorname{ch}(2\gamma(t)x - \mu(t))}.$$

If we set $x = 0$ and $x = 1$ in the last equality, then taking into account (3.2), we have the following problem

$$\begin{cases} \mu'(t) = 8\gamma^3(t) - \frac{16it\rho(t)\gamma^2(t)\omega_1(t)e^{\mu(t)}}{e^{2\mu(t)}+4\gamma^2(t)} - \frac{16\gamma^3(t)\alpha(t)e^{2\gamma(t)+\mu(t)}}{e^{2\mu(t)}+4\gamma^2(t)e^{4\gamma(t)}}, \\ \mu(0) = 0. \end{cases}$$

Solution of this problem has the form $\mu(t) = 10t$. As a result, the solution of the considered problem is expressed as follows:

$$u(x,t) = \frac{-4\gamma^2(t)}{(2\gamma^2(t)-\frac{1}{2})e^{-10t+2\gamma(t)x} + \operatorname{ch}(2\gamma(t)x - 10t)},$$
$$f_{11}(x,t) = \frac{8\gamma^2(t)e^{-(2\gamma(t)+0.5)x+10t}}{(2\gamma(t)+1)(4\gamma^2(t)+e^{-4\gamma(t)x+20t})},$$
$$f_{12}(x,t) = e^{-\frac{x}{2}} - \frac{4\gamma(t)e^{-4\gamma(t)x-0.5x+20t}}{(2\gamma(t)+1)(4\gamma^2(t)+e^{-4\gamma(t)x+20t})},$$
$$g_{11}(x,t) = te^{\frac{x}{2}+\int_0^x A(s,t)\,ds}\int_0^x \sigma(s,t)e^{-\frac{s}{2}-\int_0^s A(z,t)\,dz}\,ds,$$

$$g_{12}(x,t) = \frac{8\gamma^3(t)e^{2\gamma(t)x-10t} - 2\gamma(t)e^{-2\gamma(t)x+10t} + 4\gamma^2(t)e^{2\gamma(t)x-10t} + e^{-2\gamma(t)x+10t}}{8\gamma^2(t)} \times$$

$$\times\, te^{\frac{x}{2}+\int_0^x A(s,t)\,ds}\int_0^x \sigma(s,t)e^{-\frac{s}{2}-\int_0^s A(z,t)\,dz}\,ds + \frac{t(2\gamma(t)+1)(4\gamma^2(t)+e^{20t-4\gamma(t)x})}{8\gamma^2(t)e^{-(2\gamma(t)+\frac{1}{2})x+10t}},$$

where

$$A(x,t) = -2\gamma(t) - 1 + \frac{4\gamma(t)}{4\gamma^2(t)e^{4\gamma(t)x-20t}+1}, \quad \sigma(x,t) = \frac{-e^{\frac{x}{2}}(2\gamma(t)+1)(4\gamma^2(t)+e^{-4\gamma(t)x+20t})}{4\gamma^2(t)+2e^{-4\gamma(t)x+20t}}.$$

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Интегрирование уравнения мКдФ с нестационарными коэффициентами и дополнительными членами в случае движущихся собственных значений

Ключевые слова: интегральное уравнение Гельфанда–Левитана–Марченко, система уравнений Дирака, решения Йоста, данные рассеяния.

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В данной работе рассматривается задача Коши для нестационарного модифицированного уравнения Кортевега–де Фриза с дополнительным членом и с самосогласованным источником в случае движущихся собственных значений. Получена эволюция данных рассеяния оператора Дирака, потенциал которого является решением нагруженного модифицированного уравнения Кортевега–де Фриза с самосогласованным источником в классе быстроубывающих функций. Приведены конкретные примеры, иллюстрирующие применение полученных результатов.

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