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COALITIONAL PARETO OPTIMAL SOLUTION OF ONE DIFFERENTIAL GAME

This paper is devoted to the differential positional coalitional games with non-transferable payoffs (games without side payments). We believe that the researches of the objection and counter-objection equilibrium for non-cooperative differential games that have been carried out over the last years allow to cover some aspects of non-transferable payoff coalitional games. In this paper we consider the issues of the internal and external stability of coalitions in the class of positional differential games. For a differential positional linear-quadratic six-player game with a two-coalitional structure, the coefficient constraints are obtained which provide an internal and external stability of the coalitional structure.

Keywords: Nash equilibrium, objections and counter-objections equilibrium, Pareto optimality, coalition.

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Introduction

By the end of the last century, four research directions had been formed in the theory of positional differential games (PDG): non-cooperative, cooperative, hierarchical and coalitional variants of games. Among coalitional games, there are games with transferable payoffs (games with side payments in which players can share their profits during the game) and with non-transferable payoffs (games without side payments when no splitting of payoffs is allowed). The coalitional games with side payments are being actively explored at the Faculties of Applied Mathematics and Management Processes of St. Petersburg State University and the Institute of Mathematics and Information Technologies of Petrozavodsk State University (Professors L. A. Petrosyan, V. V. Mazalov, E. M. Parilina, A. N. Rettieva and their numerous domestic and foreign followers) [1–6]. The theory of coalitional PDG without side payments is just beginning its formation on the basis of the objections and counter-objections equilibrium; this theory is being investigated at the Department of Optimal Control of the Faculty of Computational Mathematics and Cybernetics of Moscow State University [7–10]. In this paper we will use this approach to investigate a coalitional six-persons PDG without side payments and with a two-coalition structure $\{K_1 = \{1, 2, 3\}, K_2 = \{4, 5, 6\}\}$.

Moreover, we propose a similar approach to the construction of optimal (in the formalized sense below!) solutions in coalitional DPGs based on the ideas of the Nash equilibrium principle and the Bellman dynamical programming method.

Recall that in 1949, a twenty-one-year-old graduate student at Princeton University John Forbes Nash proposed in his dissertation the concept of solving a non-cooperative game, later called *Nash equilibrium (NE)* which is a crucial concept in non-cooperative games and their applications in various sciences (mathematical economics, sociology, systems analysis, and military sciences). For his work, Nash was one of the recipients (together with Harshanyi and Selten) of the Nobel Memorial Prize in Economic Sciences in 1994. Opening almost any modern journal on game theory, operations research, systems analysis and mathematical economics now, we will almost certainly meet with papers that touch on certain issues related to Nash equilibrium. However, “there are spots on the sun”. These “spots” may be the following ones: internal and external instability of a set of Nash equilibrium situations; instability with respect to two or more players deviations from the equilibrium (NE is stable with respect to the deviation of only one of the players); NE may not exist; improvability; absence of equivalence and interchangeability; etc.

In these cases, the authors see [9] two ways out. First, limit yourself to mathematical models that are free of some of the listed negative properties. Second, introduce new concepts of equilibrium other than NE. Here, in our opinion, the equilibrium of objections and counter-objections [7, 8] and the Berge equilibrium [9, 10] are promising. In addition, in this paper we use Nash ideas to formalize a Pareto solution for *coalitional* PDGs.

We consider a non-cooperative game in normal form described by the triple:

$$\Gamma = \langle \mathbb{N},\ \{X_i\}_{i\in\mathbb{N}},\ \{f_i(x)\}_{i\in\mathbb{N}} \rangle.$$

Here $\mathbb{N} = \{1, \ldots, N\}$ is the set of players' numbers, the set of strategies x_i of the player i is $X_i \subset \mathbb{R}^{n_i}$. The players choose their strategies $x_i \in X_i$ $(i \in \mathbb{N})$ simultaneously. As a result, we get a *strategy profile* $x = (x_1, \ldots, x_N) \in X = \prod\limits_{i\in\mathbb{N}} X_i$. The aims (interests) of the players are determined by the values (*payoffs*) of payoff functions $f_i(x)$ $(i \in \mathbb{N})$. For every player i, his objective point in the game Γ is to choose his strategy so that his payoff will be as *large* as possible.

D e f i n i t i o n 0.1. A pair $(x^e, f^e = f(x^e)) \in X \times \mathbb{R}^N$ is called a *Nash equilibrium* of the game Γ if N equations

$$\max_{x_i\in X_i} f_i(x^e \| x_i) = f_i(x^e) \quad (i \in \mathbb{N}) \tag{1}$$

take place. Here we use the generally accepted in game theory designations

$$(x^e \| x_i) = \left(x_1^e, \ldots, x_{i-1}^e, x_i, x_{i+1}^e, \ldots, x_N^e\right).$$

Equations (1) imply immediately three important conditions of Nash equilibrium (NE). *First*, NE is *stable* under a deviation of a separate player from it. *Second*, NE satisfies the property of *individual rationality*, i. e.,

$$f_i(x^e) \geqslant \max_{x_i\in X_i} \min_{x_{-i}\in X_{-i}} f_i(x_i, x_{-i}) \quad (i \in \mathbb{N})$$

(here $-i = \mathbb{N} \setminus \{i\} = \{1, \ldots, i-1, i+1, \ldots, N\}$). *Third*, in the case of a zero-sum game (i. e., when in Γ the set of player's numbers is $\mathbb{N} = \{1, 2\}$ and $f_1(x) = -f_2(x) = f(x)$), x^e coincides with the *saddle point* $(x_1^e, x_2^e) \in X_1 \times X_2$ determined by the chain of equalities $\max\limits_{x_1\in X_1} f(x_1, x_2^e) = f(x_1^e, x_2^e) = \min\limits_{x_2\in X_2} f(x_1^e, x_2)$. Moreover, Definition 0.1 immediately answers two questions: 1) how should player $i \in \mathbb{N}$ act in the game? (the answer: to use $x_i^e \in X_i$); 2) what kind of payoff will he get? (the answer: $f_i(x^e)$).

Let also the game Γ be placed into the correspondence to the N*-criterion problem*

$$\Gamma_v = \langle X,\ \{f_i(x)\}_{i\in\mathbb{N}} \rangle.$$

Here the set X of *alternatives* x coincides with the set of strategy profiles of the *game* Γ, the criterion $f_i(x)$ coincides with the scalar payoff function $f_i(x)$ of the player $i \in \mathbb{N}$.

D e f i n i t i o n 0.2 (see [11–13]). An alternative $x^P \in X$ is called a *Pareto maximal alternative* in the problem Γ_v, if for any $x \in X$ the system of N inequalities $f_i(x) \geqslant f_i(x^P)$, $i \in \mathbb{N}$, is incompatible, besides at least one inequality is strict. The pair $\left(x^P, f^P = f(x^P)\right) \in X \times \mathbb{R}^N$ is called a *Pareto maximum* of the problem Γ_v; recall that $f = (f_1, \ldots, f_N) \in \mathbb{R}^N$.

It follows immediately from Definition 0.2 that when using an alternative other than x^P: 1) it is impossible to increase all criteria $f_i(x^P)$ $(i \in \mathbb{N})$ at the same time; 2) if at least one of the components $f_i(x^P)$ of the vector $f(x^P)$ increases, then at least one of the others will inevitably decrease. Moreover, Karlin 's Lemma [14] is obvious:

P r o p e r t y 0.1. *If there exist constants* $\alpha_i > 0$ $(i \in \mathbb{N})$ *such that*

$$\max_{x \in X} \sum_{i \in \mathbb{N}} \alpha_i f_i(x) = \sum_{i \in \mathbb{N}} \alpha_i f_i(x^P), \tag{2}$$

then x^P *is a Pareto maximal alternative for the problem* Γ_v.

We designate the operation of Pareto maximum construction (2) as

$$\mathrm{MAX}^P_{x \in X} f(x) = f(x^P) = f^P,$$

i. e.,

$$\mathrm{MAX}^P_{x \in X} f(x) = \max_{x \in X} \alpha' f(x) = \alpha' f(x^P) \tag{3}$$

for some constant N-vector $\alpha = (\alpha_1, \ldots, \alpha_N)$, $\alpha_i > 0$ $(i \in \mathbb{N})$; the prime means transposition (α' is row N-vector).

§ 1. Basic concepts of coalitional game theory

Here we move on to the possible version of coalitional game Γ. Let a *coalition structure* be given on the set $\mathbb{N}$. A coalition structure is a partition of the set $\mathbb{N}$ into pairwise disjoint subsets (*coalitions*) of $\mathbb{N}$, the union of which equals $\mathbb{N}$. We have restricted ourselves to the two coalitions $K_1 = \{1, 2, 3\}$ and $K_2 = \{4, 5, 6\}$ for the game Γ, $\mathbb{N} = K_1 \cup K_2$ and $K_1 \cap K_2 = \varnothing$. Players within their coalition K_l $(l = 1, 2)$ have the possibility to jointly choose their strategy $x_{K_l} = \{x_i | \, i \in K_l\} \in X_{K_l} = \prod\limits_{i \in K_l} X_i$. The set of all such strategies x_{K_l} is designated as X_{K_l}. Then every strategy profile $x \in X$ of the game Γ can be written as $x = (x_{K_1}, x_{K_2})$. Payoff vector-function of coalition K_l is designated as $f_{K_l}(x_{K_1}, x_{K_2}) = (f_m(x_{K_1}, x_{K_2}) \, | \, m \in K_l)$ $(l = 1, 2)$, so the payoff N-vector function (a vector criterion of the problem Γ_v) is $f(x) = f(x_{K_1}, x_{K_2}) = (f_{K_1}(x_{K_1}, x_{K_2}), f_{K_2}(x_{K_1}, x_{K_2}))$.

As a result, we move from the original non-coalition version of the game Γ to the coalitional game

$$G = \langle \mathbb{N} = \{K_1 \cup K_2\}, \ \{K_l\}_{l=1,2}, \ \{X_{K_l}\}_{l=1,2}, \ \{f_{K_l}(x_{K_1}, x_{K_2})\}_{l=1,2} \rangle \, .$$

The players of a separate coalition cooperatively choose a coalition strategy, fulfilling two requirements: individual and collective rationality.

The *individual rationality* condition means that the strategy profile x^P provides for the ith player a payoff which is not less than his maximin payoff, namely

$$f_i(x^P) \geqslant \max_{x_i \in X_i} \min_{x_{-i} \in X_{-i}} f_i(x_i, x_{-i}) = \min_{x_{-i} \in X_{-i}} f_i(x_i^g, x_{-i}) = f_i^g \leqslant f_i(x_i^g, x_{-i}) \; \forall x_{-i} \in X_{-i}, \; i \in \mathbb{N},$$

where $-i = \mathbb{N} \setminus \{i\} = \{1, \ldots, i-1, i+1, \ldots, N\}$, $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \in X_{-i} = \prod\limits_{j \in \mathbb{N} \setminus \{i\}} X_j$.

Note that, for the class of linear-quadratic games considered in this paper, such maximins do not exist [8] and therefore we do not take into account the conditions of individual rationality.

Turn to the *collective rationality* condition. For the members of a separate coalition, for example K_1, it comes down to the Pareto maximality (in relation to the partners from this coalition K_1), namely

$$\mathrm{MAX}^P_{x_{K_1} \in X_{K_1}} f_{K_1}(x_{K_1}, x^P_{K_2}) = f_{K_1}(x^P_{K_1}, x^P_{K_2}).$$

Thus, we come to the following definition.

D e f i n i t i o n 1.1. A strategy profile $x^P = (x^P_{K_1}, x^P_{K_2}) \in X = X_{K_1} \times X_{K_2}$ is called *coalitional Pareto-optimal* (CPO) for the game G if

$$\begin{cases} \mathrm{MAX}^P_{x_{K_1} \in X_{K_1}} f_{K_1}(x_{K_1}, x^P_{K_2}) = f_{K_1}(x^P_{K_1}, x^P_{K_2}), \\ \mathrm{MAX}^P_{x_{K_2} \in X_{K_2}} f_{K_2}(x^P_{K_1}, x_{K_2}) = f_{K_2}(x^P_{K_1}, x^P_{K_2}). \end{cases} \tag{4}$$

It is easy to see that (4) is a modification of (1) for the case of singleton coalitions in Γ (the operation $\max\limits_{x_i \in X_i}$ from (1) is replaced to the operation of Pareto maximum construction $\mathrm{MAX}^P_{x_{K_l} \in X_{K_l}}$ from (3) and (4) is a modification of NE). Naturally, the "sun spots" mentioned above, characteristic of NE, also take place for CPO.

In our opinion, Definition 1.1 is no less promising for research than Definition 0.1. However, next we will focus on the issues of internal and external stability of coalitions in PDG.

§ 2. Internal and external stability of coalition

Let $(x^P_{K_1}, x^P_{K_2}) = x^P$ be a coalitional Pareto-optimal (CPO) strategy profile (determined by (4)) and the players have decided to stick to this strategy in the coalitional game G. The reasons for this choice, for example, for the coalition K_1 are:

first, $x^P_{K_1}$ is a Pareto maximal alternative of the problem $G_1 = \langle X_{K_1},\ f_{K_1}(x_{K_1}, x^P_{K_2}) \rangle$ (players from K_1 strive to choose their strategies so that for everyone his payoff function value will be as large as possible and in the multicriteria problem G_1 the strategy $x^P_{K_1}$ provides the Pareto maximum for $f_{K_1}(x_{K_1}, x^P_{K_2})$);

second, the requirement of *internal stability* of K_1. We call K_1 an internally stable coalition if none of its players has a desire to leave K_1: either go to the coalition K_2, or form a new third coalition consisting of only one "defector". Let's assume that in K_1 at least one of the remaining players has the opportunity to "punish the defector". Formally, we define the process of punishment as follows.

Let player 1 have an *objection to the internal stability* of K_1, i. e., he has a strategy $x^T_1 \in X_1$ such that

$$f_1(x^T_1, x^P_2, x^P_3, x^P_{K_2}) > f_1(x^P_{K_1}, x^P_{K_2}). \tag{5}$$

In respond to this objection, one of the remaining in K_1 players, for instance, player 2 has a *counter-objection* if he has a strategy $x^C_2 \in X_2$ for which two inequalities

$$f_1(x^T_1, x^C_2, x^P_3, x^P_{K_2}) < f_1(x^P_{K_1}, x^P_{K_2}), \tag{6}$$

$$f_2(x^T_1, x^C_2, x^P_3, x^P_{K_2}) \geq f_2(x^T_1, x_2, x^P_3, x^P_{K_2}) \ \forall x_2 \in X_2. \tag{7}$$

are satisfied.

The first of them "nullifies" the effect of the objection because (6) reduces the payoff of the "threatening" player 1 to less than it was $f_1(x^P) = f_1(x^P_{K_1}, x^P_{K_2})$. The second inequality (7) even "pushes" player 2 to use x^C_2 because as a result, player 2 will achieve the biggest payoff he can only dream of. Similarly, the counter-objection of player 3 in response to the objection of player 1 to the internal stability of K_1 is determined, as well as the reaction of the two remaining players to the desire of the one player from the coalition K_1 to leave this coalition.

D e f i n i t i o n 2.1. The coalition K is called *internally stable* if, in response to the possibility of any player of the coalition K to leave K, at least one of the remaining players has a counter-objection (of the form (6) and (7)).

Note that the absence of objections leads, of course, to the uselessness of counter-objections.

Let's move on to the *external stability of the coalition* (for example, K_1 in the game G). Assume that the unwillingness of any player from K_2 to leave the coalition K_2 and join K_1 characterizes the external stability of K_1. It is also obvious that the internal stability of K_2 "provides" external stability of K_1 and vice versa.

Thus, the internal stability of each coalition in the coalition structure guarantees internal and external stability, which leads to the stability of the coalition structure, i. e., to the unwillingness to break the existing division of players into pairwise disjoint subsets.

Finally, we note that we achieve the fulfillment of inequalities (6) and (7) for the game PDG discussed later in Section 3 by special coefficient restrictions on the payoff functions of the players from K_1.

The further material of the paper is devoted to the construction of an explicit form of CPO (determined by 1.1) for a quite general class of PDG.

§ 3. Differential linear-quadratic six-player game

We consider a differential linear-quadratic six-player game described by

$$\Gamma_D = \left\langle \mathbb{N},\, \{K_1 = \{1,2,3\},\, K_2 = \{4,5,6\}\}\,,\, \Sigma_x,\, \{\overline{\mathfrak{U}}_i\}_{i\in\mathbb{N}},\, \{J_i(U,t_0,x_0)\}_{i\in\mathbb{N}} \right\rangle, \tag{8}$$

Here $\mathbb{N} = \{1,2,3,4,5,6\}$ is the set of players; a coalition structure (the division of $\mathbb{N}$ into pairwise disjoint subsets: $\mathbb{N} = K_1 \cup K_2 \wedge K_1 \cap K_2 = \varnothing$) is given; a controlled dynamic system Σ_x is linear (in x and u_i $(i \in \mathbb{N})$):

$$\dot{x} = A(t)x + \sum_{i\in\mathbb{N}} u_i, \quad x(t_0) = x_0,$$

the game is finished at the moment $\vartheta > 0$ and ϑ is fixed; *the game functioning interval* $t \in [t_0, \vartheta]$, $0 \leqslant t_0 \leqslant t \leqslant \vartheta$; elements of matrix $A(t)$ of dimensions $n \times n$ are assumed to be continuous on $[0,\vartheta]$ (this fact will be indicated by $A(\cdot) \in C_{n\times n}[0,\vartheta]$); $x \in \mathbb{R}^n$ is an n-dimensional state vector; a pair $(t,x) \in [t_0,\vartheta] \times \mathbb{R}^n$ is a position of the game; the initial position is (t_0, x_0); a strategy of player i is $u_i \in \mathbb{R}^n$ $(i \in \mathbb{N})$; since $u = (u_1,\ldots,u_6) \in \mathbb{R}^{6n}$ then the coalition strategies are $u_{K_1} = (u_1,u_2,u_3)$ and $u_{K_2} = (u_4,u_5,u_6)$, hence $u = (u_{K_1}, u_{K_2})$; the set of strategies of player $i \in \mathbb{N}$ is (according to [15])

$$\overline{\mathfrak{U}}_i = \{U_i \div u_i(t,x) = Q_i(t)x \mid \forall Q_i(\cdot) \in C_{n\times n}[0,\vartheta]\},$$

the strategy profile is $U = (U_1,\ldots,U_6) \in \overline{\mathfrak{U}} = \prod\limits_{i\in\mathbb{N}} \overline{\mathfrak{U}}_i$, $\overline{\mathfrak{U}}_{K_l} = \prod\limits_{j\in K_l} \overline{\mathfrak{U}}_j$ $(l = 1,2)$. A play of the game (8) is organized as follows. Each player chooses and uses his strategy $U_i \div u_i(t,x) = Q_i(t)x$ (i. e., uses his specific matrix $Q_i(\cdot) \in C_{n\times n}[0,\vartheta]$). Then the solution $x(t)$, $t \in [0,\vartheta]$, is constructed for the system of homogeneous and linear differential equations with continuous (in t) coefficients

$$\dot{x}(t) = \left[A(t) + \sum_{i\in\mathbb{N}} Q_i(t)\right] x, \quad x(t_0) = x_0.$$

By means of this solution *the realizations* of the strategies $u_i[t] = u_i(t, x(t)) = Q_i(t)x(t)$ $(i \in \mathbb{N})$ chosen by the players are formed. Note, that n-vectors $u_i[t]$ are continuous on $[t_0,\vartheta]$. On such a continuous pairs $(x(t), u[t] = (u_1[t],\ldots,u_6[t]))$ the *payoff function* of player i is a priori defined as a quadratic functional

$$J_i\left(U,t_0,x_0\right) = x'(\vartheta)\overline{C}_i x(\vartheta) + \int_{t_0}^{\vartheta} \left(\sum_{j\in\mathbb{N}} u_j'[t]\overline{D}_{ij}u_j[t]\right) dt \quad (i \in \mathbb{N}), \tag{9}$$

the prime means transposition, the matrices $\overline{C}_i$ and $\overline{D}_{ij}$ of dimensions $n \times n$ are assumed to be symmetric without loss of generality. Note, in (9) the first term is called a *terminal term* and the second one is called an *integral term*. The value of (9) is called the *payoff* of player i in the game Γ_D. *In terms of "meaning"*, the players within each coalition cooperatively choose their strategies so that the components of their three-coordinate payoffs $J_{K_l} = (J_r | r \in K_l)$ $(l = 1, 2)$ will be as large as possible (and satisfy the condition of individual rationality). When choosing the optimal solution, we will use the coalitional Pareto-maximal strategy profile (see Definition 1.1).

Firstly, we simplify the controlled system of Γ_D using the substitution $y = X^{-1}(t)x$ where the matrix $X(t)$ of dimensions $n \times n$ represents the fundamental system of solutions for the equation $\dot{x} = A(t)x$, $X(\vartheta) = E_n$ (E_n is the identity matrix of dimensions $n \times n$). As a result, the system Σ_x turns into Σ_y:

$$\frac{dy}{dt} = \sum_{i \in \mathbb{N}} u_i, \quad y(t_0) = X^{-1}(t_0)x_0,$$

the set $\overline{\mathfrak{U}_i}$ of strategies of player i turns into

$$\mathfrak{U}_i = \{U_i \div u_i(t, y) = Q_i(t)y \mid \forall Q_i(\cdot) \in C_{n \times n}[0, \vartheta]\},$$

the payoff function $J_i(U, t_0, x_0)$ of the ith player turns into

$$\mathfrak{J}_i(U, t_0, y_0) = y'(\vartheta)C_i y(\vartheta) + \int_{t_0}^{\vartheta} \left(\sum_{j \in \mathbb{N}} u_j'[t] D_{ij} u_j[t] \right) dt \quad (i \in \mathbb{N}), \tag{10}$$

where the constant matrices C_i, D_{ij} of dimensions $n \times n$ are symmetric.

As a result, game (8) is reduced to the form

$$\Gamma_d = \langle \mathbb{N}, \{K_1, K_2\}, \Sigma_y, \{\mathfrak{U}_i\}_{i \in \mathbb{N}}, \{\mathfrak{J}_i(U, t_0, y_0)\}_{i \in \mathbb{N}} \rangle. \tag{11}$$

Let's give a possible economic interpretation for (11). Suppose there is an industrial cluster consisting of six companies that are, in addition, in two associations. As a rule, the company's goal is to simultaneously reduce costs ($C_i < 0$) and increase internal investment ($D_{ii} > 0$) in its own production. An additional condition is the opposite interests of the other cluster members (if $D_{ij} < 0$ $(i \neq j)$).

In view of this interpretation, we assume that

$$C_i < 0, \ D_{ii} > 0, \ D_{ij} < 0 \quad (i, j \in \mathbb{N};\ i \neq j). \tag{12}$$

Now we should apply Definition 1.1 to the differential game (11). Namely, for each coalition K_1 and K_2 we introduce a set of its strategies $U_{K_l} \in \mathfrak{U}_{K_l} = \prod_{r \in K_l} \mathfrak{U}_r$ $(l = 1, 2)$. Besides we use a three-dimensional functional of its payoffs, which, in view of $U = (U_{K_1}, U_{K_2})$, is of the form $\mathfrak{J}_{K_l} = (\mathfrak{J}_j | j \in K_l)$ $(l = 1, 2)$. Then

$$\mathfrak{J}_{K_1}(U, t_0, y_0) = (\mathfrak{J}_1(U_{K_1}, U_{K_2}, t_0, y_0), \mathfrak{J}_2(U_{K_1}, U_{K_2}, t_0, y_0), \mathfrak{J}_3(U_{K_1}, U_{K_2}, t_0, y_0))$$

and

$$\mathfrak{J}_{K_2}(U, t_0, y_0) = (\mathfrak{J}_4(U_{K_1}, U_{K_2}, t_0, y_0), \mathfrak{J}_5(U_{K_1}, U_{K_2}, t_0, y_0), \mathfrak{J}_6(U_{K_1}, U_{K_2}, t_0, y_0)).$$

D e f i n i t i o n 3.1. A pair $(U^P; \mathfrak{J}^P) = \left(U^P_{K_1}, U^P_{K_2}; \mathfrak{J}^P_{K_1}(U^P, t_0, y_0), \mathfrak{J}^P_{K_2}(U^P, t_0, y_0)\right) \in \mathfrak{U} \times \mathbb{R}^6$ is called a coalitional Pareto-optimal solution (CPOS) of the game Γ_d if *for all initial positions* $(t_0, y_0) \in [t_0, \vartheta] \times \mathbb{R}^n$, $y_0 \neq 0_n$,

$$\begin{cases} \mathrm{MAX}^P_{U_{K_1} \in \mathfrak{U}_{K_1}} \mathfrak{J}_{K_1}(U_{K_1}, U^P_{K_2}, t_0, y_0) = \mathfrak{J}^P_{K_1}(U^P, t_0, y_0), \\ \mathrm{MAX}^P_{U_{K_2} \in \mathfrak{U}_{K_2}} \mathfrak{J}_{K_2}(U^P_{K_1}, U_{K_2}, t_0, y_0) = \mathfrak{J}^P_{K_2}(U^P, t_0, y_0), \end{cases}$$

where, for example, $\mathrm{MAX}^P_{U_{K_1}\in\mathfrak{U}_{K_1}}\mathfrak{J}_{K_1}(U_{K_1}, U^P_{K_2}, t_0, y_0)$ means a Pareto maximality of the three-dimensional functional $\mathfrak{J}_{K_1}(U_{K_1}, U^P_{K_2}, t_0, y_0)$ on the set $\mathfrak{U}_{K_1}$.

In this paper the Pareto maximum will be realized by following Property 0.1 (by finding the scalar maximum for the linear convolution of the three components $\mathfrak{J}_{K_1}(U_{K_1}, U^P_{K_2}, t_0, y_0)$ with positive coefficients).

§ 4. Auxiliary assertions from the theory of matrices and quadratic forms

Further, for a constant and symmetric matrix D of dimensions $n\times n$, the inequality $D > 0$ (< 0) means that the quadratic form $x'Dx$ is positive definite (negative definite), where $x\in\mathbb{R}^n$.

P r o p o s i t i o n 4.1 (see [16, p. 108]). *The two chains of implications:*
a) $D > 0 \Rightarrow 0 \leqslant \lambda x'x \leqslant x'Dx \leqslant \Lambda x'x\ \forall x\in\mathbb{R}^n$;
b) $D < 0 \Rightarrow -\Lambda x'x \leqslant x'Dx \leqslant -\lambda x'x\ \forall x\in\mathbb{R}^n$;
take place. Here $\lambda(-\Lambda)$ *is the smallest root and* $\Lambda(-\lambda)$ *is the largest root of the characteristic equation* $\det[D-\lambda E_n] = 0$; $0 < \lambda \leqslant \Lambda$, E_n *is the identity matrix of dimensions* $n\times n$.

P r o p o s i t i o n 4.2. *Let* Λ *be the largest root of the characteristic equation* $\det[D-\lambda E_n] = 0$ *and* $D > 0$. *Then*
a) $\Lambda < nM$, *where* M *is a maximum of modules of elements* d_{ij} *of matrix* $D = (d_{ij})$ [16];
b) $\Lambda < \min\limits_{i=1,\dots,n}\sum\limits_{j=1}^{n}|d_{ij}|$ [17].

P r o p o s i t i o n 4.3. *The equivalence* $D < 0 \Leftrightarrow (-1)D = -D > 0$ *is valid* (*here we multiply all the elements of constant symmetric* $n\times n$*-matrix* D *by minus one*) *and then the largest root* $-\Lambda > 0$ *of the characteristic equation* $\det[-D-\lambda E_n] = 0$ *coincides with the smallest root of the characteristic equation* $\det[D-\lambda E_n] = 0$.

R e m a r k 4.1. According to Proposition 4.3 to estimate the smallest root of the characteristic $\det[D-\lambda E_n] = 0$ it is sufficient to estimate the largest root of the characteristic equation $\det[-D-\lambda E_n] = 0$.

P r o p o s i t i o n 4.4 (the analogue of Lemmas 4.1 and 4.2 from [9]). *The following implications are valid* ($i, j\in\mathbb{N}$, $i\neq j$):

(a) $D_{ii} > 0 \Rightarrow$ *for every* $U^*_{-i}\in\mathfrak{U}_{-i}$ *and* $U^*_i\in\mathfrak{U}_i$ *there exists its own constant*

$$\alpha^*_i(U^*_i, U^*_{-i}, t_0, y_0) > 0$$

such that for all constants $\alpha > \alpha^*_i(U^*_i, U^*_{-i})$ *and for the strategy* $\overline{U_i} \div \alpha y$ *the strict inequality*

$$\mathfrak{J}_i(\overline{U_i}, U^*_{-i}, t_0, y_0) > \mathfrak{J}_i(U^*_i, U^*_{-i}, t_0, y_0)$$

is valid. Recall that the payoff function $\mathfrak{J}_i$ *is determined in* (10) *and*

$$-i\in\mathbb{N}\setminus\{i\} = \{1,\dots,i-1,i+1,\dots,N\};$$

(b) $D_{ij} < 0$ $(i\neq j) \Rightarrow$ *for all* $U^*_j\in\mathfrak{U}_j$ *and* $U^*_{-j}\in\mathfrak{U}_{-j}$ *there exists its own constant*

$$\alpha^*_j(U^*_j, U^*_{-j}, t_0, y_0) > 0$$

such that for $\forall\alpha > \alpha^*_j(U^*_j, U^*_{-j})$ *and* $\overline{U_j}\div\alpha y$ *we get*

$$\mathfrak{J}_j(\overline{U_j}, U^*_{-j}, t_0, y_0) < \mathfrak{J}_j(U^*_j, U^*_{-j}, t_0, y_0).$$

And lastly, the following propositions are established in [7, 8].

T h e o r e m 4.1. *If* (12) *is satisfied for the game* Γ_d, *then:*

a) *a Nash equilibrium does not exist;*

b) $\min\limits_{U_i \in \mathfrak{U}_i} \mathfrak{J}_i(U_i, U_{-i}, t_0, y_0)$ *does not exist, and that is precisely why, when determining the optimal solution of the game* Γ_d, *the condition of individual rationality can be ignored;*

c) *if, in addition to* (12), *restrictions on the roots of the corresponding characteristic equations* $\Lambda_{11}\Lambda_{22} < \Lambda_{12}\Lambda_{21}$ *are satisfied, then in the game* (11) *there exists* [7] *a Pareto equilibrium of objections and counter-objections.*

In conclusion, we turn to the central result of this paper: the construction of an explicit form of a CPO-solution for coalitional game (11). To do this we will use Property 0.1 and Bellman's dynamical programming method. It will also be necessary to solve one static N-criterion problem, with which the next section begins.

§ 5. Pareto maximal strategy profiles and Pareto payoffs

Let us set out some auxiliary assertions (see Lemma 5.1 below).

Consider a static six-criterion problem

$$\Gamma_6 = \left\{\mathbb{R}^{6n},\ \{f_i(u) = u_1' D_{i1} u_1 + \ldots + u_6' D_{i6} u_6\}_{i=1,\ldots,6}\right\},$$

in the problem Γ_6 the decision maker's aim is to choose an alternative $u = (u_1, \ldots, u_6) \in \mathbb{R}^{6n}$ so that the values of all 6 components of the vector criterion $f(u) = (f_1(u), \ldots, f_6(u))$ will be as large as possible. Here the analogue of Definition 0.2 is the following: an alternative u^P is *Pareto maximal* for the game Γ_6 if for all $u \in \mathbb{R}^{6n}$ the system of inequalities $f_i(u) \geqslant f_i(u^P)$ $(i = 1, \ldots, 6)$, where at least one inequality is strict, is incompatible.

Below we use the analogue of Property 0.1.

L e m m a 5.1. *Let in* Γ_6 *the constant matrices* D_{ij} *of dimensions* $n \times n$ *be symmetric, and the positive numbers* Λ_{ii}, Λ_{ij} $(i, j = 1, \ldots, 6,\ i \neq j)$ *satisfy the inequalities*

$$D_{ii} > 0,\ D_{ij} < 0\ (i \neq j), \quad \Lambda_{11}\Lambda_{22} < \Lambda_{12}\Lambda_{21}, \quad \Lambda_{44}\Lambda_{55} < \Lambda_{45}\Lambda_{54}.$$

Then for constants α_i^* $(i \in \mathbb{N})$ *such that*

$$\begin{aligned} &\alpha_1^* = 1, \qquad \alpha_2^* = \frac{1}{2}\left(\frac{\Lambda_{11}}{\Lambda_{21}} + \frac{\Lambda_{12}}{\Lambda_{22}}\right), \qquad \alpha_3^* = \frac{1}{2}\left(\frac{\Lambda_{13} + \alpha_2^*\Lambda_{23}}{\Lambda_{33}}\right), \\ &\alpha_4^* = 1, \qquad \alpha_5^* = \frac{1}{2}\left(\frac{\Lambda_{44}}{\Lambda_{54}} + \frac{\Lambda_{45}}{\Lambda_{55}}\right), \qquad \alpha_6^* = \frac{1}{2}\left(\frac{\Lambda_{46} + \alpha_5^*\Lambda_{45}}{\Lambda_{66}}\right), \end{aligned} \tag{13}$$

the quadratic forms

$$\begin{aligned} f(u) = \alpha_1^* f_1(u) + \alpha_2^* f_2(u) + \alpha_3^* f_3(u) + \alpha_4^* f_4(u) + \alpha_5^* f_5(u) + \alpha_6^* f_6(u) = \\ = u_1' D_1(\alpha^*) u_1 + \ldots + u_6' D_6(\alpha^*) u_6 \end{aligned}$$

are negative definite. Here

$$D_i(\alpha^*) = \alpha_1^* D_{1i} + \alpha_2^* D_{2i} + \alpha_3^* D_{3i} + \alpha_4^* D_{4i} + \alpha_5^* D_{5i} + \alpha_6^* D_{6i},$$

besides, $\Lambda_{ii} > 0$ *is the largest root of the characteristic equation* $\Delta_{ii}(\Lambda) = \det[D_{ii} - \Lambda E_n] = 0$ *and* $-\Lambda_{ij} < 0$ *is the largest by absolute value root of the equation* $\delta_{ij}(\Lambda) = \det[D_{ij} - \Lambda E_n] = 0$, $(i, j \in \{1, \ldots, 6\},\ j \neq i)$.

P r o o f. Due to the matrices $D_{ii} > 0,\ D_{ij} < 0\ (i, j \in \mathbb{N};\ i \neq j)$ of dimensions $n \times n$ are symmetric, the roots of the characteristic equations $\Delta_{ii}(\Lambda) = 0$ and $\delta_{ij}(\Lambda) = 0$ are real and $\Lambda_{ii} > 0,\ -\Lambda_{ij} < 0\ (i, j \in \mathbb{N},\ i \neq j)$. Since $u_i' D_{ii} u_i \leqslant \Lambda_{ii} u_i' u_i$ and $u_j D_{ij} u_j' \leqslant -\Lambda_{ij} u_j' u_j$ (see Proposition 4.1) then we write

$$f(u) = \alpha_1^* f_1(u) + \alpha_2^* f_2(u) + \ldots + \alpha_6^* f_6(u) =$$

$$= u_1'[\alpha_1^* D_{11} + \alpha_2^* D_{21} + \ldots + \alpha_6^* D_{61}]u_1 + \ldots + u_6'[\alpha_1^* D_{16} + \alpha_2^* D_{26} + \ldots + \alpha_6^* D_{66}]u_6 \leqslant$$

$$\leqslant [\alpha_1^*\Lambda_{11} + \alpha_2^*(-\Lambda_{21}) + \ldots + \alpha_6^*(-\Lambda_{61})]u_1'u_1 + \ldots + [\alpha_1^*(-\Lambda_{16}) + \alpha_2^*(-\Lambda_{26}) + \ldots + \alpha_6^*(+\Lambda_{66})]u_6'u_6.$$

The components α_i^* of the vector-column $\alpha^* = (\alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^* \alpha_5^*, \alpha_6^*)$ are given in (13). Since $u_i' D_{ii} u_i \leqslant \Lambda_{ii} u_i' u_i$ and $u_j' D_{ij} u_j \leqslant -\Lambda_{ij} \|u_j\|^2$ the scalar function $f(u) < 0$ for all $u \in \mathbb{R}^{6n}$, $u \neq 0_{6n}$, if all the inequalities from Table 1 are satisfied.

Table 1

$\Lambda_{11}\alpha_1^* - \Lambda_{21}\alpha_2^* - \Lambda_{31}\alpha_3^* - \Lambda_{41}\alpha_4^* - \Lambda_{51}\alpha_5^* - \Lambda_{61}\alpha_6^* < 0$
$-\Lambda_{12}\alpha_1^* + \Lambda_{22}\alpha_2^* - \Lambda_{32}\alpha_3^* - \Lambda_{42}\alpha_4^* - \Lambda_{52}\alpha_5^* - \Lambda_{62}\alpha_6^* < 0$
$-\Lambda_{13}\alpha_1^* - \Lambda_{23}\alpha_2^* + \Lambda_{33}\alpha_3^* - \Lambda_{43}\alpha_4^* - \Lambda_{53}\alpha_5^* - \Lambda_{63}\alpha_6^* < 0$
$-\Lambda_{14}\alpha_1^* - \Lambda_{24}\alpha_2^* - \Lambda_{34}\alpha_3^* + \Lambda_{44}\alpha_4^* - \Lambda_{54}\alpha_5^* - \Lambda_{64}\alpha_6^* < 0$
$-\Lambda_{15}\alpha_1^* - \Lambda_{25}\alpha_2^* - \Lambda_{35}\alpha_3^* - \Lambda_{45}\alpha_4^* + \Lambda_{55}\alpha_5^* - \Lambda_{65}\alpha_6^* < 0$
$-\Lambda_{16}\alpha_1^* - \Lambda_{26}\alpha_2^* - \Lambda_{36}\alpha_3^* - \Lambda_{46}\alpha_4^* - \Lambda_{56}\alpha_5^* + \Lambda_{66}\alpha_6^* < 0$

Moreover, we get for

$$\begin{cases} \Lambda_{11}\alpha_1^* - \Lambda_{21}\alpha_2^* - \Lambda_{31}\alpha_3^* < 0, \\ -\Lambda_{12}\alpha_1^* + \Lambda_{22}\alpha_2^* - \Lambda_{32}\alpha_3^* < 0, \\ -\Lambda_{13}\alpha_1^* - \Lambda_{23}\alpha_2^* + \Lambda_{33}\alpha_3^* < 0, \end{cases} \tag{14}$$

and

$$\begin{cases} \Lambda_{44}\alpha_4^* - \Lambda_{54}\alpha_5^* - \Lambda_{64}\alpha_6^* < 0, \\ -\Lambda_{45}\alpha_4^* + \Lambda_{55}\alpha_5^* - \Lambda_{65}\alpha_6^* < 0, \\ -\Lambda_{46}\alpha_4^* - \Lambda_{56}\alpha_5^* + \Lambda_{66}\alpha_6^* < 0, \end{cases} \tag{15}$$

that all 6 strict inequalities from Table 1 take place, because with the exception of (14) and (15), all the other terms are negative (since $-\Lambda_{ij} < 0$, $\alpha_i^* > 0$, $i \neq j$).

We establish that $\Lambda_{11}\Lambda_{22} < \Lambda_{12}\Lambda_{21}$ and $\Lambda_{44}\Lambda_{55} < \Lambda_{45}\Lambda_{54}$ yield the fulfilment of the first two inequalities from Table 1. Really, if $\alpha_3^* > 0$ and $0 < \dfrac{\Lambda_{11}}{\Lambda_{21}} < \alpha_2^* < \dfrac{\Lambda_{12}}{\Lambda_{22}}$ (that follows immediately from $\Lambda_{11}\Lambda_{22} < \Lambda_{12}\Lambda_{21}$) then the first two strict inequalities (14) take place. Finally, the third inequality from (14) is valid for $0 < \alpha_3^* < \dfrac{1}{2}\dfrac{\Lambda_{13} + \Lambda_{23}\alpha_2^*}{\Lambda_{33}}$, where $\alpha_2^* = \dfrac{1}{2}\left(\dfrac{\Lambda_{11}}{\Lambda_{21}} + \dfrac{\Lambda_{12}}{\Lambda_{22}}\right)$. Similarly, $\alpha_4^* = 1$, $\alpha_5^* = \dfrac{1}{2}\left(\dfrac{\Lambda_{44}}{\Lambda_{54}} + \dfrac{\Lambda_{45}}{\Lambda_{55}}\right)$, $\alpha_6^* = \dfrac{1}{2}\left(\dfrac{\Lambda_{46} + \alpha_5^*\Lambda_{45}}{\Lambda_{66}}\right)$ imply the fulfilment of (15). □

P r o p o s i t i o n 5.1. *If, in the differential game* Γ_d,

$$D_{ii} > 0,\ D_{ij} < 0,\ C_i < 0\ (i, j = 1, \ldots, 6;\ i \neq j),\ \Lambda_{11}\Lambda_{22} < \Lambda_{12}\Lambda_{21},\ \Lambda_{44}\Lambda_{55} < \Lambda_{45}\Lambda_{54}, \tag{16}$$

then, for the 6*-criterion problem a Pareto maximum strategy profile* U^P*, we have*

$$\begin{aligned} U^P = \left(U_1^P, U_2^P, \ldots, U_6^P\right) \div \left(u_1^P(t,y), u_2^P(t,y), \ldots, u_6^P(t,y)\right) = \\ = u^P(t,y) = \left(Q_1^P(t)y, Q_2^P(t)y, \ldots, Q_6^P(t)y\right) = \\ = \left(-D_1^{-1}(\alpha^*)\Theta^P(t)y, -D_2^{-1}(\alpha^*)\Theta^P(t)y, \ldots, -D_6^{-1}(\alpha^*)\Theta^P(t)y\right), \end{aligned} \tag{17}$$

where $\Theta^P(\cdot)$ *is a symmetric and continuous on* $[0, \vartheta]$ *matrix of dimensions* $n \times n$

$$\Theta^P(t) = \left\{C^{-1}(\alpha^*) + \int_t^{\vartheta} \left[D_1^{-1}(\alpha^*) + D_2^{-1}(\alpha^*) + \ldots + D_6^{-1}(\alpha^*)\right] \, d\tau\right\}^{-1} \tag{18}$$

and constant symmetric $n \times n$*-matrices*

$$D_i(\alpha^*) = \alpha_1^* D_{1i} + \alpha_2^* D_{2i} + \ldots + \alpha_6^* D_{6i} \ (i = 1, \ldots, 6), \tag{19}$$

where positive numbers $\alpha_1^*, \alpha_2^*, \ldots, \alpha_6^*$ *are determined in Lemma* 5.1.

P r o o f. We will find the Pareto maximum strategy profile U^P by applying Lemma 5.1. We will specifically use Table 1 and the dynamical programming method (DPM) (see [12, p. 112]). The application of DPM here includes two stages as follows. In the *first stage*, we will find six positive numbers $\alpha_1^*, \alpha_2^*, \ldots, \alpha_6^*$ and a continuously differentiable scalar function $V(t, y) = = y'\Theta(t)y$, $\Theta(t) = \Theta'(t)$ $\forall t \in [0, \vartheta]$ and n-dimensional vector functions $u_i(t, y, V)$ $(i \in \mathbb{N})$ such that for all $y \in \mathbb{R}^n$

$$V(\vartheta, y) = y'C(\alpha^*)y, \quad C(\alpha^*) = \alpha_1^* C_1 + \alpha_2^* C_2 + \ldots + \alpha_6^* C_6.$$

Using the scalar function

$$W(t, y, u_1, \ldots, u_6, V) = \frac{\partial V}{\partial t} + \left[\frac{\partial V}{\partial y}\right]' (u_1 + \ldots + u_6) + \alpha_1^* u_1' D_1(\alpha^*) u_1 + \ldots + \alpha_6^* u_6' D_6(\alpha^*) u_6,$$

in view of $\left(\dfrac{\partial V}{\partial y} = \operatorname{grad}_y V\right)$ and

$$\max_{u_1, \ldots, u_6} W(t, y, u_1, \ldots, u_6, V) = \operatorname{Idem}\{u_i \to u_i(t, y, V) \ (i = 1, \ldots, 6)\} \tag{20}$$

for all $(t, y, V) \in [0, \vartheta] \times \mathbb{R}^{n+1}$, we will determine n-dimensional vector functions $u_i(t, y, V)$ $(i \in \mathbb{N})$. The fulfilment of

$$\begin{aligned} \left.\frac{\partial W}{\partial u_i}\right|_{u(t,y,V)} &= \frac{\partial V}{\partial y} + 2D_i(\alpha^*) u_i(t, y, V) = 0_n \quad (i = 1, \ldots, 6), \\ \frac{\partial^2 W}{\partial u_i^2} &= 2D_i(\alpha^*) < 0 \quad (i = 1, \ldots, 6), \end{aligned} \tag{21}$$

(due to Lemma 5.1, $D_i(\alpha^*) < 0$) for all $(t, y) \in [0, \vartheta) \times \mathbb{R}^n$, is a sufficient condition for existence of $u(t, y, V)$ in (20).

From (21), we get

$$u_i(t, y, V) = -\frac{1}{2} D_i^{-1}(\alpha^*) \frac{\partial V}{\partial y} \quad (i = 1, \ldots, 6). \tag{22}$$

Then

$$W(t,y,u(t,y,V),V) = W[t,y,V] = \frac{\partial V}{\partial t} - \frac{1}{4}\left[\frac{\partial V}{\partial y}\right]' \left(D_1^{-1}(\alpha^*) + \ldots + D_6^{-1}(\alpha^*)\right)\frac{\partial V}{\partial y}.$$

Second stage. We will solve a partial differential equation

$$W(t,y,V) = 0$$

under the boundary-value condition $(C(\alpha^*) = \alpha_1^* C_1 + \alpha_2^* C_2 + \ldots + \alpha_6^* C_6)$

$$V(\vartheta, y) = y'C(\alpha^*)y \quad \forall y \in \mathbb{R}^n.$$

The solution $V = V^P(t,y)$ is constructed in the class of the quadratic forms $V^P(t,y) = y'\Theta^P(t)y$ with a matrix $\Theta^P(t) = \left[\Theta^P(t)\right]'$ of dimensions $n \times n$. Then, for all $t \in [0,\vartheta]$ and for all $y \in \mathbb{R}^n$, we get

$$W\left[t,y,V(t,y) = y'\Theta^P y\right] = 0, \quad V(\vartheta,y) = y'C(\alpha^*)y \quad \forall y \in \mathbb{R}^n.$$

Both of these requirements will hold if the symmetric matrix $\Theta^P(t)$ of dimensions $n\times n$ is a solution of the matrix differential equation of Riccati type ($0_{n\times n}$ is a null matrix of dimensions $n \times n$):

$$\dot{\Theta}^P(t) - \Theta^P(t)\left(D_1^{-1}(\alpha^*) + \ldots + D_6^{-1}(\alpha^*)\right)\Theta^P(t) = 0_{n\times n},$$

$$\Theta^P(\vartheta) = C(\alpha^*) = \alpha_1^* C_1 + \alpha_2^* C_2 + \ldots + \alpha_6^* C_6.$$

The solution $\Theta^P(t)$ of this matrix equation is of the form (18) [12, p. 65]. Here we take into account the implication

$$C_i < 0 \ (i = 1,\ldots,6) \Rightarrow C(\alpha^*) = \alpha_1^* C_1 + \alpha_2^* C_2 + \ldots + \alpha_6^* C_6 < 0.$$

Finally, the validity of (17) follows from (22). Thus the Pareto maximum strategy profile U^P is of the form (17)–(19). □

Now, we will construct the Pareto maximum payoffs

$$J^P = \left(J_1^P,\ldots,J_6^P\right) = \left(J_1(U^P,t_0,y_0),\ldots,J_6(U^P,t_0,y_0)\right)$$

using the dynamical programming method [12].

P r o p o s i t i o n 5.2. *Let requirements* (16) *from Proposition* 5.1 *be fulfilled, and, in the game* Γ_d*, let six scalar continuous differentiable functions* $V_i(t,y) = y'\Theta_i(t)y$ $(i = 1,\ldots,6)$ *be found such that*

1) $V_i(\vartheta,y) = y'C_i y \ \forall y \in \mathbb{R}^n$*;*

2) *the system of six partial differential equations*

$$\begin{aligned} &\frac{\partial V_i}{\partial t} + \left[\frac{\partial V_i}{\partial y}\right]' N(t)y + y'\Theta^P(t)M_i(t)\Theta^P(t)y = 0, \\ &V_i(\vartheta,y) = y'C_i y \quad \forall y \in \mathbb{R}^n \quad (i = 1,\ldots,6) \end{aligned} \tag{23}$$

has a solution which is of the form $V_i(t,y) = y'\Theta_i(t)y$, $[\Theta_i(t)]' = \Theta_i(t)$ $(i = 1,\ldots,6)$.

Then, for any initial position $(t_0,y_0) \in [0,\vartheta) \times \mathbb{R}^n$*,* $y_0 \neq 0_n$*, we have*

$$J_i^P = J_i(U^P,t_0,y_0) = y_0'\Theta_i(t_0)y_0 \quad (i = 1,\ldots,6).$$

In (23)*, the continuous matrices of dimensions* $n \times n$

$$N(t) = -\left(D_1^{-1}(\alpha^*) + \ldots + D_6^{-1}(\alpha^*)\right)\Theta^P(t),$$

$$M_i(t) = \Theta^P(t)\left[D_1^{-1}(\alpha^*)D_{i1}D_1^{-1}(\alpha^*) + \ldots + D_6^{-1}(\alpha^*)D_{i6}D_6^{-1}(\alpha^*)\right]\Theta^P(t) \quad (i = 1, \ldots, 6),$$

matrices $\Theta^P(t)$ and $D_i(\alpha^*)$ of dimensions $n \times n$ are given in (18) *and* (19)*, symmetric $n \times n$ matrices*

$$\Theta_i(t) = \left[Y^{-1}(t)\right]'\left\{C_i - \int_t^{\vartheta} Y'(\tau)\Theta^P(\tau)M_i(\tau)\Theta^P(\tau)Y(\tau)\,d\tau\right\}Y^{-1}(t) \quad (i = 1, \ldots, 6), \quad (24)$$

$Y(t)$ is a fundamental matrix of solutions for the homogeneous system $\dot{y} = N(t)y$, $y(\vartheta) = E_n$.

P r o o f. We construct the scalar functions

$$\begin{aligned} W_i[t, y, V_i] =& \frac{\partial V_i}{\partial t} + \left[\frac{\partial V_i}{\partial y}\right]' N(t)y + \left[u_1^P(t, y)\right]' D_{i1}u_1^P(t, y) + \ldots + \\ &+ \left[u_6^P(t, y)\right]' D_{i6}u_6^P(t, y) \quad (i = 1, \ldots, 6), \end{aligned} \quad (25)$$

where n-dimensional vector functions $u_i^P(t, y)$ are determined in (17).

Next, we solve the system of six partial differential equations

$$W_i[t, y, V_i] = 0, \quad V_i(\vartheta, y) = y'C_i y \quad \forall y \in \mathbb{R}^n \quad (i = 1, \ldots, 6). \quad (26)$$

The solution $V_i(t, y)$ $(i = 1, \ldots, 6)$ of (26) is constructed in the class of the quadratic forms $V_i(t, y) = y'\Theta_i(t)y$, $[\Theta_i(t)]' = \Theta_i(t)$ $(i = 1, \ldots, 6)$.

We will set up two facts.

First, the solution of system (25) and (26) satisfies the equality

$$V_i(t_0, y_0) = J_i(U^P, t_0, y_0) \quad (i = 1, \ldots, 6), \quad (27)$$

where the strategy profile $U^P = (U_1^P, \ldots, U_6^P)$ is of the form (17). Actually, if U^P is a strategy profile from (16)–(19), then, in view of (25) and (26), for the solution $y^P(t)$ of system $\dot{y} = N(t)y$, $y(t_0) = y_0 \neq 0_n$, and also $y = y^P(t)$, we get

$$0 = W_i[t, y^P(t), V_i(t, y^P(t))] = \frac{\partial V_i(t, y^P(t))}{\partial t} + \left[\frac{\partial V_i(t, y^P(t))}{\partial y}\right]' N(t)y^P(t) +$$

$$+ \sum_{j=1}^{6}\left[u_j^P(t, y^P(t))\right]' D_{ij}u_j^P(t, y^P(t)) = W_i[t] \quad \forall t \in [t_0, \vartheta] \ (i = 1, \ldots, 6).$$

Integrating both sides of this equality from t_0 to ϑ and using the boundary-value condition from (26) we obtain

$$0 = \int_{t_0}^{\vartheta} W_i[t]\,dt = \int_{t_0}^{\vartheta} \frac{dV_i(t, y^P(t))}{dt}\,dt + \int_{t_0}^{\vartheta}\sum_{j=1}^{6}\left[u_j^P(t, y^P(t))\right]' D_{ij}u_j^P(t, y^P(t))\,dt =$$

$$= V_i(\vartheta, y^P(\vartheta)) - V_i(t_0, y^P(t_0)) + \int_{t_0}^{\vartheta}\sum_{j=1}^{6}\left[u_j^P(t, y^P(t))\right]' D_{ij}u_j^P(t, y^P(t))\,dt =$$

$$= y'(\vartheta)C_i y(\vartheta) + \int_{t_0}^{\vartheta}\sum_{j=1}^{6}\left[u_j^P(t, y^P(t))\right]' D_{ij}u_j^P(t, y^P(t))\,dt - V_i(t_0, y^P(t_0)) =$$

$$= J_i(U^P, t_0, y_0) - V_i(t_0, y^P(t_0)) \quad (i = 1, \ldots, 6),$$

this result finally proves (27).

Second, we will establish that the solution $V_i(t,y)$ $(i \in \mathbb{N})$ of system (26) has the form $V_i(t,y) = y'\Theta_i(t)y$, where the symmetric matrix $\Theta_i(t)$ of dimensions $n \times n$ can be represented as (24). Actually, substituting $V_i(t,y) = y'\Theta_i(t)y$ into (26), we see that (24) will be valid if $\Theta_i(t)$ $(i = 1,\ldots,6)$ is a solution of the linear inhomogeneous matrix differential equation

$$\dot{\Theta}_i + \Theta_i N + N\Theta_i + \Theta^P(t)M_i\Theta^P(t) = 0_{n\times n}, \quad \Theta_i(\vartheta) = C_i \quad (i = 1,\ldots,6). \tag{28}$$

Substituting $\Theta_i(t)$ from (24) into (28), we will make sure that the symmetric matrix $\Theta_i(t)$ of dimensions $n \times n$ is really the solution of (28). This completes the proof of Proposition 5.2. $\square$

R e m a r k 5.1. Combining Propositions 5.1 and 5.2 leads to the following end result concerning the explicit form of the Pareto maximum solution $(U^P, J^P) \in \mathfrak{U} \times \mathbb{R}^6$ for the game Γ_d.

Let in the game Γ_d:

1) constant symmetric matrices of dimensions $n \times n$ satisfy

$$D_{ii} > 0,\ D_{ij} < 0,\ C_i < 0 \quad (i,j = 1,\ldots,6;\ i \neq j);$$

2) $\Lambda_{11}\Lambda_{22} < \Lambda_{12}\Lambda_{21},\ \Lambda_{44}\Lambda_{55} < \Lambda_{45}\Lambda_{54}$.

Then for all $(t_0, y_0) \in [0,\vartheta) \times \mathbb{R}^n$, $y_0 \neq 0_n$, we have

$$U^P \div u^P(t,y) = \left(-D_1^{-1}(\alpha^*)\Theta^P(t)y, -D_2^{-1}(\alpha^*)\Theta^P(t)y, \ldots, -D_6^{-1}(\alpha^*)\Theta^P(t)y\right),$$

$$J^P = (J_1^P, J_2^P, \ldots, J_6^P), \quad J_i^P = y_0'\Theta_i(t_0)y_0 \quad (i = 1,\ldots,6),$$

and the symmetric matrices $\Theta_i^P(t)$ and $\Theta_i(t)$ of dimensions $n \times n$ are of the form:

$$\Theta^P(t) = \left\{C^{-1}(\alpha^*) + \int_t^{\vartheta}\left[D_1^{-1}(\alpha^*) + D_2^{-1}(\alpha^*) + \ldots + D_6^{-1}(\alpha^*)\right] d\tau\right\}^{-1},$$

$$\Theta_i(t) = \left[Y^{-1}(t)\right]'\left\{C_i - \int_t^{\vartheta} Y'(\tau)\Theta^P(\tau)M_i(\tau)\Theta^P(\tau)Y(\tau)\,d\tau\right\}Y^{-1}(t) \quad (i = 1,\ldots,6),$$

$n \times n$-matrix $Y(t)$ is a fundamental matrix of solutions for the homogeneous system $\dot{y} = N(t)y$, $Y(\vartheta) = E_n$, the symmetric matrices

$$C(\alpha^*) = \alpha_1^*C_1 + \alpha_2^*C_2 + \ldots + \alpha_6^*C_6, \quad D_i(\alpha^*) = \alpha_1^*D_{1i} + \alpha_2^*D_{2i} + \ldots + \alpha_6^*D_{6i},$$

$$N(t) = -\left(D_1^{-1}(\alpha^*) + \ldots + D_6^{-1}(\alpha^*)\right)\Theta^P(t),$$

$$M_i(t) = \Theta^P(t)\left[D_1^{-1}(\alpha^*)D_{i1}D_1^{-1}(\alpha^*) + \ldots + D_6^{-1}(\alpha^*)D_{i6}D_6^{-1}(\alpha^*)\right]\Theta^P(t),$$

the positive numbers $\alpha_1^*, \alpha_2^*, \ldots, \alpha_6^*$ are defined by a recurrent way

$$\alpha_1^* = 1, \quad \alpha_2^* = \frac{1}{2}\left(\frac{\Lambda_{11}}{\Lambda_{21}} + \frac{\Lambda_{12}}{\Lambda_{22}}\right), \quad \alpha_3^* = \frac{1}{2}\left(\frac{\Lambda_{13} + \alpha_2^*\Lambda_{23}}{\Lambda_{33}}\right),$$

$$\alpha_4^* = 1, \quad \alpha_5^* = \frac{1}{2}\left(\frac{\Lambda_{44}}{\Lambda_{54}} + \frac{\Lambda_{45}}{\Lambda_{55}}\right), \quad \alpha_6^* = \frac{1}{2}\left(\frac{\Lambda_{46} + \alpha_5^*\Lambda_{45}}{\Lambda_{66}}\right),$$

where Λ_{ii} $(-\Lambda_{ij})$ is the largest (the smallest) root of the characteristic equation $\det[D_{ii} - \Lambda E_n] = 0$ (respectively $\det[D_{ij} - \Lambda E_n] = 0$) $(i,j \in \{1,\ldots,6\},\ i \neq j)$.

§ 6. Explicit form of CPOS

Let's move on to the main result of the present paper. We construct the explicit form of coalitional Pareto optimal solution (CPOS) for the game Γ_d. According to Definition 1.1, for the game Γ_d, under restrictions (16) the following equalities are valid:

$$\begin{cases} \mathrm{MAX}^P_{U_{K_1}\in\mathfrak{U}_{K_1}}\mathfrak{J}_{K_1}(U_{K_1},U^P_{K_2},t_0,y_0)=\mathfrak{J}_{K_1}(U^P,t_0,y_0),\\ \mathrm{MAX}^P_{U_{K_2}\in\mathfrak{U}_{K_2}}\mathfrak{J}_{K_2}(U^P_{K_1},U_{K_2},t_0,y_0)=\mathfrak{J}_{K_2}(U^P,t_0,y_0),\end{cases}$$

which will follow from

$$\begin{cases} \max\limits_{U_{K_1}\in\mathfrak{U}_{K_1}}\left[\alpha_1^*\mathfrak{J}_1(U_{K_1},U^P_{K_2},t_0,y_0)+\alpha_2^*\mathfrak{J}_2(U_{K_1},U^P_{K_2},t_0,y_0)+\alpha_3^*\mathfrak{J}_3(U_{K_1},U^P_{K_2},t_0,y_0)\right]=\\ \qquad=\sum\limits_{j=1}^{3}\alpha_j^*\mathfrak{J}_j(U^P,t_0,y_0),\\ \max\limits_{U_{K_2}\in\mathfrak{U}_{K_2}}\left[\alpha_4^*\mathfrak{J}_4(U^P_{K_1},U_{K_2},t_0,y_0)+\alpha_5^*\mathfrak{J}_5(U^P_{K_1},U_{K_2},t_0,y_0)+\alpha_6^*\mathfrak{J}_6(U^P_{K_1},U_{K_2},t_0,y_0)\right]=\\ \qquad=\sum\limits_{m=4}^{6}\alpha_m^*\mathfrak{J}_m(U^P,t_0,y_0),\end{cases} \tag{29}$$

for all $(t_0,y_0)\in[0,\vartheta)\times\mathbb{R}^n$ where the constants

$$\alpha_1^*=1,\quad \alpha_2^*=\frac{1}{2}\left(\frac{\Lambda_{11}}{\Lambda_{21}}+\frac{\Lambda_{12}}{\Lambda_{22}}\right),\quad \alpha_3^*=\frac{1}{2}\left(\frac{\Lambda_{13}+\alpha_2^*\Lambda_{23}}{\Lambda_{33}}\right),$$

$$\alpha_4^*=1,\quad \alpha_5^*=\frac{1}{2}\left(\frac{\Lambda_{44}}{\Lambda_{54}}+\frac{\Lambda_{45}}{\Lambda_{55}}\right),\quad \alpha_6^*=\frac{1}{2}\left(\frac{\Lambda_{46}+\alpha_5^*\Lambda_{45}}{\Lambda_{66}}\right),$$

$\Lambda_{ii}>0$ is the largest root of $\Delta_{ii}(\Lambda)=\det[D_{ii}-\Lambda E_n]=0$, $-\Lambda_{ij}<0$ is the smallest root of the equation $\delta_{ij}(\Lambda)=\det[D_{ij}-\Lambda E_n]=0$, $(i,j\in\{1,\dots,6\},\ j\neq i)$, $\Lambda_{ij}>0$; a strategy profile $U^P=(U_1^P,\dots,U_6^P)\div\big(-D_1^{-1}(\alpha^*)\Theta^P(t)y,\dots,-D_6^{-1}(\alpha^*)\Theta^P(t)y\big)$.

So, we show that the strategy profile $U^P\in\mathfrak{U}$ found in the previous section of the present paper is just a combination $(U^P_{K_1},U^P_{K_2})=U^P$, where $U^P_{K_1}=(U_1^P,U_2^P,U_3^P)$ and $U^P_{K_2}=(U_4^P,U_5^P,U_6^P)$ are found in (17)–(19). The proof of the validity of (29) is presented in article [7] for the game Γ_d, where the strategies $U^P_{K_2}=(U_4^P,U_5^P,U_6^P)\in\mathfrak{U}_{K_2}$ are frozen (see Proposition 3.1 from [7] for $\alpha^*=1$, $\beta=\alpha_2^*$, $\gamma=\alpha_3^*$). Moreover, these $(U_1^P,U_2^P,U_3^P)=U^P_{K_1}$ just realize $\max\limits_{U_{K_1}\in\mathfrak{U}_{K_1}}$ in (29) and this fact in combination with Property 0.1 implies that $(U^P_{K_1},U^P_{K_2})=U^P$ is Pareto maximal for the three-criterion problem $\Big\langle \dot y=u_1+u_2+u_3,\ y(t_0)=y_0,\ \mathfrak{U}_{K_1},\ \big\{\mathfrak{J}_i(U_1,U_2,U_3,U^P_{K_2},t_0,y_0)\big\}_{i=1,2,3}\Big\rangle$. Hence, the explicit form of $U^P_{K_1}$ is $U^P_{K_1}=(U_1^P,U_2^P,U_3^P)\div\big(-D_1^{-1}(\alpha^*)\Theta^P(t)y,-D_2^{-1}(\alpha^*)\Theta^P(t)y,-D_3^{-1}(\alpha^*)\Theta^P(t)y\big)$ and the corresponding payoffs can be found for $U^P_{K_1}\in\mathfrak{U}_{K_1}$. The validity of $U^P_{K_2}=$ $=(U_4^P,U_5^P,U_6^P)\div\big(-D_4^{-1}(\alpha^*)\Theta^P(t)y,-D_5^{-1}(\alpha^*)\Theta^P(t)y,-D_6^{-1}(\alpha^*)\Theta^P(t)y\big)$ and the explicit form of payoffs (for $U^P_{K_2}=(U_4^P,U_5^P,U_6^P)$ are established in the same way. Thus, we obtain the validity of the following theorem.

T h e o r e m 6.1. *Let for the differential coalitional game with non-transferable payoffs*

$$\Gamma_d=\langle\{K_1=\{1,2,3\},\ K_2=\{4,5,6\}\},\ \Sigma_y,\ \{\mathfrak{U}_{K_l}\}_{l=1,2},\ \{\mathfrak{J}_{K_l}(U_{K_1},U_{K_2},t_0,y_0)\}_{l=1,2}\rangle$$

the following restrictions be satisfied:

$$D_{ii}>0,\ D_{ij}<0,\ C_i<0\quad (i,j=1,\dots,6;\ i\neq j);\ \Lambda_{11}\Lambda_{22}<\Lambda_{12}\Lambda_{21},\ \Lambda_{44}\Lambda_{55}<\Lambda_{45}\Lambda_{54}.$$

Then for the game Γ_d the coalitional Pareto optimal solution is formed by the quadruple

$$\big(U^P_{K_1},\, U^P_{K_2};\ \mathfrak{J}_{K_1}(U^P_{K_1}, U^P_{K_2}, t_0, y_0),\ \mathfrak{J}_{K_2}(U^P_{K_1}, U^P_{K_2}, t_0, y_0)\big) \in \mathfrak{U}_{K_1} \times \mathfrak{U}_{K_2} \times \mathbb{R}^3 \times \mathbb{R}^3,$$

where $U^P_{K_1} = (U^P_1, U^P_2, U^P_3) \in \mathfrak{U}_{K_1}$, $U^P_{K_2} = (U^P_4, U^P_5, U^P_6) \in \mathfrak{U}_{K_2}$, $U^P_i \div -D_i(\alpha^*)\Theta^P(t)y$ $(i = 1, \ldots, 6)$ *are symmetric constant matrices of dimensions* $n \times n$,

$$D_i(\alpha^*) = \alpha^*_1 D_{1i}(\alpha^*) + \alpha^*_2 D_{2i}(\alpha^*) + \alpha^*_3 D_{3i}(\alpha^*) + \alpha^*_4 D_{4i}(\alpha^*) + \alpha^*_5 D_{5i}(\alpha^*) + \alpha^*_6 D_{6i}(\alpha^*)$$
$$(i = 1, \ldots, 6),$$

$$\alpha^*_1 = 1, \qquad \alpha^*_2 = \frac{1}{2}\bigg(\frac{\Lambda_{11}}{\Lambda_{21}} + \frac{\Lambda_{12}}{\Lambda_{22}}\bigg), \qquad \alpha^*_3 = \frac{1}{2}\bigg(\frac{\Lambda_{13} + \alpha^*_2\Lambda_{23}}{\Lambda_{33}}\bigg),$$

$$\alpha^*_4 = 1, \qquad \alpha^*_5 = \frac{1}{2}\bigg(\frac{\Lambda_{44}}{\Lambda_{54}} + \frac{\Lambda_{45}}{\Lambda_{55}}\bigg), \qquad \alpha^*_6 = \frac{1}{2}\bigg(\frac{\Lambda_{46} + \alpha^*_5\Lambda_{45}}{\Lambda_{66}}\bigg),$$

continuous symmetric matrix of dimensions $n \times n$

$$\Theta^P(t) = \bigg\{C^{-1}(\alpha^*) + \int_t^{\vartheta} \big[D_1^{-1}(\alpha^*) + D_2^{-1}(\alpha^*) + \ldots + D_6^{-1}(\alpha^*)\big]\, d\tau\bigg\}^{-1},$$

$$\mathfrak{J}^P_{K_1}[t_0, y_0] = \mathfrak{J}_{K_1}(U^P_{K_1}, U^P_{K_2}, t_0, y_0) =$$
$$= \big(\mathfrak{J}_1(U^P_{K_1}, U^P_{K_2}, t_0, y_0), \mathfrak{J}_2(U^P_{K_1}, U^P_{K_2}, t_0, y_0), \mathfrak{J}_3(U^P_{K_1}, U^P_{K_2}, t_0, y_0)\big),$$
$$\mathfrak{J}^P_{K_2}[t_0, y_0] = \mathfrak{J}_{K_2}(U^P_{K_1}, U^P_{K_2}, t_0, y_0) =$$
$$= \big(\mathfrak{J}_4(U^P_{K_1}, U^P_{K_2}, t_0, y_0), \mathfrak{J}_5(U^P_{K_1}, U^P_{K_2}, t_0, y_0), \mathfrak{J}_6(U^P_{K_1}, U^P_{K_2}, t_0, y_0)\big),$$

$$\mathfrak{J}^P_{K_1} = (y_0'\Theta_1(t_0)y_0, y_0'\Theta_2(t_0)y_0, y_0'\Theta_3(t_0)y_0)\,,\ \mathfrak{J}^P_{K_2} = (y_0'\Theta_4(t_0)y_0, y_0'\Theta_5(t_0)y_0, y_0'\Theta_6(t_0)y_0)\,,$$

$$C(\alpha^*) = \sum_{i=1}^{6} \alpha^*_i C_i,$$

$$\Theta_i(t) = \big[Y^{-1}(t)\big]' \bigg[C_i - \int_t^{\vartheta} Y'(\tau)\Theta^P(\tau)M_i(\tau)\Theta^P(\tau)Y(\tau)\, d\tau\bigg] Y^{-1}(t) \quad (i = 1, \ldots, 6),$$

$Y(t)$ *is a fundamental matrix of solutions for the system* $\dot{y} = N(t)y$, $Y(\vartheta) = E_n$,

$$N(t) = -\sum_{i=1}^{6} D_i^{-1}(\alpha^*)\Theta^P(t), \quad M_i(t) = \Theta^P(t)\left[\sum_{j=1}^{6} D_j^{-1}(\alpha^*)D_{ij}D_j^{-1}(\alpha^*)\right],$$

where $\Lambda_{ii} > 0$ *is the largest root of* $\Delta(\lambda)\det[D_{ii} - \Lambda E_n] = 0$, $-\Lambda_{ij} < 0$ *is the smallest root of the equation* $\delta_{ij}(\Lambda) = \det[D_{ij} - \Lambda E_n] = 0$ $(i, j \in \{1, \ldots, 6\}, i \neq j)$.

In this case, both coalitions are internally and externally stable.

P r o o f. It was established in [7] that if $D_{11} > 0$ then a Nash equilibrium does not exist in the game Γ_d, but due to Proposition 5.1 from [7] there may be an objection to the internal stability of coalition K_1 (i. e., there are exists $U^T_1 \div \alpha y$ and $\alpha^* = \text{const} > 0$ such that for all $\alpha > \alpha^*$ we have $\mathfrak{J}_1(U^T_1, U^P_2, U^P_3, U^P_{K_2}, t_0, y_0) > \mathfrak{J}_1(U^P, t_0, y_0)$).

Due to Proposition 5.4 from [7] and $D_{12} < 0$, in response to the objection, the player 2 from coalition K_1 must use $\bar{\alpha_1} > 0$ such that for all $\alpha > \bar{\alpha_1}$ and $U^C_2 \div \alpha y$ we have

$$\mathfrak{J}_1(U^T_1, U^C_2, U^P_3, U^P_{K_2}, t_0, y_0) < \mathfrak{J}_1(U^P, t_0, y_0).$$

Similarly, in view of $D_{22} > 0$, there exists a number $\bar{\alpha_2} > 0$ such that for all $\alpha > \bar{\alpha_2}$ we have

$$\mathfrak{J}_2(U^T_1, U^C_2, U^P_3, U^P_{K_2}, t_0, y_0) \geq \mathfrak{J}_2(U^T_1, U_2, U^P_3, U^P_{K_2}, t_0, y_0) \quad \forall U_2 \in \mathfrak{U}_2.$$

But then, for the strategy U^C_2 and $\alpha > \max\{\bar{\alpha_1}, \bar{\alpha_2}\}$, the last two strict inequalities are combined into a counter-objection to the internal stability of coalition K_1 by player 1.

Thus, we have established the internal stability of K_1. The internal stability of K_2 is established in the same way. □

Conclusion

Now we have established that, if restrictions (16) are satisfied, then in the game Γ_d there exists a coalitional Pareto-optimal solution (its explicit form can be found in Theorem 6.1). At the end of the paper, we would like to mention that the techniques proposed here can be used to investigate the stability of other coalition structures.

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В. И. Жуковский, Л. В. Жуковская, С. Н. Сачков, Е. Н. Сачкова
Коалиционное Парето-оптимальное решение одной дифференциальной игры

Ключевые слова: равновесие по Нэшу, равновесие угроз и контругроз, оптимальность по Парето, коалиция.

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Статья посвящена дифференциальным позиционным коалиционным играм с нетрансферабельными выигрышами (играм без побочных платежей). Авторы надеются, что исследования равновесия угроз и контругроз для бескоалиционных игр, проведенные в последние годы, позволят охватить некоторые аспекты коалиционных игр с нетрансферабельными выигрышами. В настоящей статье мы рассматриваем вопросы внутренней и внешней устойчивости коалиций для класса позиционных дифференциальных игр. Для дифференциальной позиционной линейно-квадратичной игры шести игроков с двухкоалиционной структурой получены коэффициентные критерии, обеспечивающие внутреннюю и внешнюю устойчивость коалиционной структуры.

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