Izvestiya Instituta Matematiki i Informatiki Udmurtskogo Gosudarstvennogo Universiteta
2024. Volume 63. Pp. 61–79

MSC2020: 47H09, 47H10

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SET-VALUED MAPPINGS: FIXED POINT RESULTS WITH β-FUNCTION AND SOME APPLICATIONS

Via the so-called β-function, some fixed-point results for nonexpansive set-valued mappings are obtained. In this study, the results are considered in the context of complete metric spaces which are neither uniformly convex nor compact. Our theorems extend, unify, and improve several recent results in the existing literature. In the end, we apply our new results to ensure the existence of a solution for a nonlinear integral inclusion. Moreover, we approximate the fixed point by a faster iterative process.

Keywords: fixed point, set-valued mappings, T_β-multivalued contractive mapping, T_β-weakly contractive multivalued mapping, integral inclusion, iterative process.

DOI: 10.35634/2226-3594-2024-63-05

Introduction

In fixed point theory for single-valued mappings, Banach contraction principle (for short, BCP) [4] (1922) is perhaps the most celebrated and useful tool in all of analysis, in particular in nonlinear analysis. It has been extended in different directions and many fixed point theorems.

In 1962, Edelstein mentioned in [9], to obtain a fixed point of strict contractive mappings on a metric space (X, d) ($d(Tx, Ty) < d(x, y)$ for all $x \neq y \in X$), it is necessary to add the compactness assumption of the space.

In 1965, Browder [6] and Göhde [12] independently showed one of the most interesting extensions of BCP by proving that every nonexpansive mapping whose Lipschitz's constant equal to 1 (that is $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in X$) of a closed convex and bounded subset of the Banach space X has a fixed point, if the subset is supposed to be uniformly convex (for each $0 < \varepsilon \leq 2$, there exists $\delta > 0$ such that for all $\|x\| \leq 1, \|y\| \leq 1$ the condition $\|x - y\| \geq \varepsilon$ implies that $\|\frac{x+y}{2}\| \leq 1 - \delta$ see [8]).

For multivalued mappings ($T\colon X \to 2^X$), fixed point theory plays a major role in various fields of pure and applied mathematics because of its many applications, for instance, game theory, real and complex analysis, optimal control problems as well as integral inclusions etc.

Nadler (1969) [20] was the first mathematician who combined the concept of contraction (see condition below) with the notion of multivalued mappings. In other words, the author in [20] proved the existence of a fixed point $u \in X$ (i. e., $u \in Tu$) on a metric space (X, d) for a multivalued mapping T satisfying the following condition

$$H(Tx, Ty) \leq kd(x, y),$$

for all $x, y \in X$ where H is the Hausdorff metric induced by the metric d.

In 2012, Samet et al. [26] introduced the concept of α-admissible functions and α-ψ-contractive mappings and established various fixed point theorems for such mappings in the setting of complete metric spaces. Asl et al. [3] extended the notion of α-admissibility to α_*-admissibility for multivalued mappings and showed that α_*-admissible function is also α-admissible, but the converse may not be true as shown in [3]. For more detail see [14, 16, 25].

On the other hand, very recently, in 2020 the authors in [29] introduced a new class of strict contraction for multivalued mappings as follows

$$\inf_{x \neq y \in X} \{d(x, y) - H(Tx, Ty)\} > 0 \tag{0.1}$$

and proved some fixed point results without the compactness assumption of the space. In this direction, recent works can be found in [28, 30–35].

Note that the fixed point theory of nonexpansive multivalued mappings is more difficult but more important than the corresponding theory of single-valued mappings and the strict class of multivalued mappings. So, it is a very natural question to ask: if we can extend (0.1) to

$$\inf_{x\neq y\in X}\{d(x,y)-H(Tx,Ty)\}\geq 0$$

and prove, in this area, a fixed point theorem for a new class of nonexpansive multivalued mappings. For further information on this topic, interested readers are directed to the latest papers [2, 13, 36].

In this paper, via the concept of α_*-admissible functions [3] and the notion of symmetric spaces discussed in [11] we give an affirmative answer to the above-asked question. In other words, we introduce the concept of T_β-contractive multivalued mappings and prove a fixed point theorem for this type of contractions which is a class of nonexpansive multivalued mappings without using neither the compactness nor the uniform convexity of the space X. Also, some examples are presented to show the importance of the proven results.

Furthermore, motivated by the notion of T-weakly contractive multivalued mappings in [29] (see also, weakly contractive maps defined in [1, 7]), we employ our first result to prove a fixed point theorem for the newly called T_β-weakly contractive multivalued mappings.

Moreover, as an application of our studies, we prove the existence of a solution for Volterra-type integral inclusion

$$x(t)\in f(t)+\int_0^t K(t,s,x(s))\,ds,\quad t\in[0,\tau], \tag{0.2}$$

where $K\colon [0,\tau]\times[0,\tau]\times\mathbb{R}\to\mathcal{P}_{cv}(\mathbb{R})$, where $\mathcal{P}_{cv}(\mathbb{R})$ denotes the class of nonempty compact and convex subsets of $\mathbb{R}$. We point out that the study of (0.2) is under new and weak conditions.

After existence theorems, it is natural to find an iteration scheme to approximate the fixed point. In this paper, we propose the following scheme:

$$\begin{cases} x_1 = x\in X,\\ z_n=(1-\beta_n)x_n+\beta_n u_n,\\ y_n=(1-\alpha_n)x_n+\alpha_n v_n,\\ x_{n+1}=w_n, & n\in\mathbb{N}, \end{cases}$$

where sequences $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $(0,1)$ such that $u_n\in P_T(x_n)$, $v_n\in P_T(z_n)$, $w_n\in P_T(y_n)$ and we denote by $P_T(x)=\{y\in Tx\colon \|x-y\|=d(x,Tx)\}$ for a multivalued mapping $T\colon X\to C(X)$, where $C(X)$ denotes the family of all closed subsets of the space (X,d).

Our scheme converges faster compared to the existing ones in the literature, which is always preferred in the practice.

§ 1. Preliminaries

The main purpose of this section is to introduce some concepts and results required in this article. We begin with the following definition.

D e f i n i t i o n 1.1 (see [37]). Let X be a nonempty set. *A symmetric* on X is a nonnegative real valued function D on $X\times X$ such that

(i) $D(x,y)=0$ if and only if $x=y$,

(ii) $D(x,y)=D(y,x)$.

A symmetric space is a pair (X,D) where D is a symmetric on X.

D e f i n i t i o n 1.2 (see [11, Definition 2.1.1]). Let (X, D) be a symmetric space and A be a nonempty subset of X.

(i) A is called *D-closed* if and only if $\overline{A}^D = A$, where

$$\overline{A}^D = \{x \in X\colon D(x, A) = 0\} \text{ and } D(x, A) = \inf\{D(x, y)\colon y \in A\}.$$

(ii) A is called *D-bounded* if and only if $\delta_D(A) < \infty$, where

$$\delta_D(A) = \sup\{D(x, y)\colon x, y \in A\}.$$

D e f i n i t i o n 1.3 (see [11, Definition 2.1.2]). Let (X, D) be a D-bounded symmetric space and let $C_D(X)$ be the class of all nonempty D-closed sets of (X, D). Consider the function $H_D\colon 2^X \times 2^X \to \mathbb{R}^+$ defined by

$$H_D(A, B) = \max\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(b, A)\},$$

for all $A, B \in C_D(X)$. H_D is called *the symmetric Hausdorff distance induced by* D.

R e m a r k 1.1. $(C_D(X), H_D)$ is a symmetric space [11, Remark 2.1.1].

Recall that:

- A sequence in a symmetric space (X, D) is called a D-Cauchy sequence if it satisfies the usual metric condition $\lim\limits_{n,m\to\infty} D(x_n, x_m) = 0$.
- A symmetric space (X, D) is called S-complete if for every D-Cauchy sequence $\{x_n\}$, there exists $x \in X$ such that $\lim\limits_{n\to\infty} D(x, x_n) = 0$.
- A symmetric space (X, D) satisfies the axiom $(W.4)$ given by Wilson [37]: if $\{x_n\} \subset X$, $\{y_n\} \subset X$ and $x \in X$ are such that $\lim\limits_{n\to\infty} D(x_n, x) = 0$ and $\lim\limits_{n\to\infty} D(x_n, y_n) = 0$, then $\lim\limits_{n\to\infty} D(y_n, x) = 0$.

We state the following result.

T h e o r e m 1.1 (see [11, Theorem 2.2.1]). *Let (X, D) be D-bounded and S-complete symmetric space satisfying $(W.4)$ and $T\colon X \to C_D(X)$ be a multivalued mapping such that*

$$H_D(Tx, Ty) \le kD(x, y), \quad k \in [0, 1), \quad \forall x, y \in X.$$

Then there exists $u \in X$ such that $u \in Tu$.

D e f i n i t i o n 1.4 (see [19]). If $\phi\colon X \to 2^Y$, then *a selection* for ϕ is a continuous mapping $f\colon X \to Y$ such that $f(x) \in \phi(x)$, where X and Y are two topological spaces.

D e f i n i t i o n 1.5 (see [19]). A mapping $T\colon X \to 2^Y$ is called *lower semicontinuous* provided that, whenever $x \in X$ and V is an open set such that $Tx \cap V \ne \emptyset$, there exists an open neighborhood U of x such that for all $y \in U$, we have $Ty \cap V \ne \emptyset$.

T h e o r e m 1.2 (Michael's selection theorem [19]). *Let X be a paracompact space and Y be a Banach space. Let $F\colon X \to 2^Y$ be a lower semicontinuous multivalued map with nonempty convex closed values. Then there exists a continuous selection $f\colon X \to Y$ of F.*

L e m m a 1.1 (see [27, Lemma 2.2]). *Let (X, d) be a metric space and $A, B \in CB(X)$, where $CB(X)$ denotes the family of all bounded and closed subsets of (X, d). If there exists $\gamma > 0$ such that*

(i) *for each $a \in A$, there is $b \in B$ such that $d(a, b) \le \gamma$,*

(ii) *for each* $b \in B$*, there is* $a \in A$ *such that* $d(b,a) \leq \gamma$,
then $H(A,B) \leq \gamma$.

In 2012, Asl et al. [3] introduced the notion of α_*-ψ-contractive multifunction type and established some fixed point theorems for these mappings. Let us denote by Ψ the family of nondecreasing functions $\psi\colon [0,\infty) \to [0,\infty)$ such that $\psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ.

D e f i n i t i o n 1.6 (see [3]). Let (X,d) be a metric space, $T\colon X \to 2^X$ be a closed-valued multifunction, $\psi \in \Psi$ and $\alpha\colon X \times X \to [0,\infty)$ be a function. We say that T is an α_*-ψ-*contractive multifunction* whenever $\alpha_*(x,y)H(Tx,Ty) \leq \psi(d(x,y))$ for $x,y \in X$, where H is the Hausdorff metric and $\alpha_*(A,B) = \inf\{\alpha(a,b)\colon a \in A,\ b \in B\}$. Also, we say that T is α_*-admissible whenever $\alpha(x,y) \geq 1$ implies $\alpha_*(Tx,Ty) \geq 1$.

T h e o r e m 1.3 (see [3]). *Let* (X,d) *be a complete metric space,* $\alpha\colon X \times X \to \mathbb{R}^+$ *be a function,* $\psi \in \Psi$ *be a strictly increasing map and* T *be a closed-valued,* α_**-admissible and* α_*-ψ*-contractive multivalued mapping on* X. *Suppose that there exist* $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $\alpha(x_0,x_1) \geq 1$. *Assume that if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n,x_{n+1}) \geq 1$ *for all* n *and* $\{x_n\}$ *converges to* x, *then* $\alpha(x_n,x) \geq 1$ *for all* $n \in \mathbb{N}$. *Then* T *has a fixed point.*

At the end of this section, we recall the proven results in [29].

L e m m a 1.2 (see [29, Lemma 2]). *Let* (X,d) *be a bounded metric space.*
(i) *Let* $D(x,y) = e^{d(x,y)} - 1$ *for all* $x,y \in X$*, then* (X,D) *is a* D*-bounded symmetric space.*
(ii) *Let* $H_D(A,B) = e^{H(A,B)} - 1$ *for all* $A,B \in C_D(X)$*, then the function* H_D *is a symmetric Hausdorff distance.*

T h e o r e m 1.4 (see [29, Theorem 3]). *Let* (X,d) *be a bounded complete metric space and let* T *be a multivalued mapping from* X *into* $C(X)$. *Suppose that*

$$\inf_{x \neq y \in X}\{d(x,y) - H(Tx,Ty)\} > 0.$$

Then T *has a fixed point.*

D e f i n i t i o n 1.7 (see [29, Definition 5]). Let (X,d) be a metric space and $T\colon X \to C(X)$ be a multivalued mapping. T will be called *a* T*-weakly contractive multivalued mapping* if for all $x,y \in X$

$$H(Tx,Ty) \leq d(x,y) - \phi[1 + d(x,y)],$$

where $\phi\colon [1,\infty) \to [0,\infty)$ is a function satisfying:
(i) $\phi(1) = 0$;
(ii) $\inf_{t>1} \phi(t) > 0$.

T h e o r e m 1.5 (see [29, Theorem 4]). *Let* $T\colon X \to C(X)$ *be a* T*-weakly contractive multivalued mapping of a bounded complete metric space* (X,d). *Then there exists* $u \in X$ *such that* $u \in Tu$.

§ 2. Main results

In this section, we prove two auxiliary lemmas that we will need for the proof of our main results.

L e m m a 2.1. *Let* (X,d) *be a bounded complete metric space. Then* (X,D) *is a* D*-bounded* S*-complete symmetric space, where* $D = e^d - 1$.

P r o o f. Let (X, d) be a complete metric space and $\{x_n\} \subset X$ a D-Cauchy sequence. Then $\lim_{n,m\to\infty} D(x_n, x_m) = 0$, and hence $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Since X is complete, there exists $u \in X$ such that $\lim_{n\to\infty} d(u, x_n) = 0$. Finally, we deduce that $\lim_{n\to\infty} D(u, x_n) = 0$. □

L e m m a 2.2. *Let (X, D) be a D-bounded and S-complete symmetric space such that $(W.4)$ is satisfied and let $T: X \to C(X)$ be an α_*-admissible mapping satisfying*

$$\alpha(x, y)H_D(Tx, Ty) \le kD(x, y),$$

for all $x, y \in X$ and for a constant $k \in [0, 1)$. Assume that there exist x_0 and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \ge 1$. Suppose that if $\{x_n\} \subset X$ is a sequence convergent to $x \in X$ such that $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \ge 1$ for all $n \in \mathbb{N}$. Then T has a fixed point; that is, $u \in Tu$.

P r o o f. We suppose that $D(x_0, x_1) > 0$ and $x_1 \notin Tx_1$, otherwise, the proof is finished. Let $a \in (k, 1)$, so

$$0 < D(x_1, Tx_1) \le \alpha(x_0, x_1)H_D(Tx_0, Tx_1) \le kD(x_0, x_1) < aD(x_0, x_1).$$

It follows that there exists $x_2 \in Tx_1$ such that

$$D(x_1, x_2) < aD(x_0, x_1).$$

Since $\alpha_*(Tx_0, Tx_1) \ge 1$, $x_1 \in Tx_0$ and $x_2 \in Tx_1$, we achieve $\alpha(x_1, x_2) \ge 1$.

If $x_2 \in Tx_2$, the proof is completed. Assume that $x_2 \notin Tx_2$, hence,

$$0 < D(x_2, Tx_2) \le \alpha(x_1, x_2)H_D(Tx_1, Tx_2) \le kD(x_1, x_2) < aD(x_1, x_2),$$

which implies that there exists $x_3 \in Tx_2$ such that

$$D(x_2, x_3) < aD(x_1, x_2).$$

Repeating this process, we get a sequence $\{x_n\} \subset X$ satisfying: $x_n \in Tx_{n-1}$, $\alpha(x_{n-1}, x_n) \ge 1$ and

$$D(x_n, x_{n+1}) < a^n D(x_0, x_1),$$

for all $n \in \mathbb{N}$. Let $n, m \in \mathbb{N}$, we obtain

$$D(x_n, x_{n+m}) < a^n D(x_0, x_m) < a^n \delta_D(X),$$

where $\delta_D(X) = \sup\{D(x, y): x, y \in X\}$. Then $\{x_n\}$ is a D-Cauchy sequence which implies that there exists $u \in X$ such that $\lim_{n\to\infty} D(u, x_n) = 0$.

Since $\alpha(x_n, u) \ge 1$, so there exists $\{y_n\} \subset Tu$ such that

$$D(x_{n+1}, Tu) \le \alpha(x_n, u)H_D(Tx_n, Tu) \le kD(x_n, u),$$

for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} D(x_{n+1}, Tu) = 0$. Then there exists $\{y_n\} \subset Tu$ such that $\lim_{n\to\infty} D(x_{n+1}, y_n) = 0$. Using $(W.4)$, we obtain $\lim_{n\to\infty} D(u, y_n) = 0$, which implies that $u \in \overline{Tu}^D = Tu$. □

Now, we introduce the notion of T_β-contractive multivalued mapping.

D e f i n i t i o n 2.1. Let T be a multivalued mapping of a bounded metric space (X, d). T is said to be *T_β-contractive multivalued mapping* if

$$\inf_{x \neq y \in X} \Big\{ d(x, y) - H(Tx, Ty) + \beta(x, y) \Big\} > 0,$$

where $\beta\colon X \times X \to \mathbb{R}$ is a function satisfying

$$\beta(x, y) \leq 0 \implies \beta^*(Tx, Ty) \leq 0,$$

with

$$\beta^*(Tx, Ty) = \sup\{\beta(a, b) \colon a \in Tx, b \in Ty\}.$$

We are ready to state and prove our main results.

T h e o r e m 2.1. *Let (X, d) be a bounded complete metric space and $T\colon X \to C(X)$ be a T_β-contractive multivalued mapping such that:*

(i) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\beta(x_0, x_1) \leq 0$;*

(ii) *if $\{x_n\} \subset X$ is a sequence convergent to $x \in X$ such that $\beta(x_n, x_{n+1}) \leq 0$ for all $n \in \mathbb{N}$, then $\beta(x_n, x) \leq 0$ for all $n \in \mathbb{N}$;*

(iii) $\beta(a, b) \leq \inf_{x \neq y \in X} \Big\{ d(x, y) - H(Tx, Ty) + \beta(x, y) \Big\}$ *for all $a, b \in X$.*

Then T has a fixed point $u \in X$.

P r o o f. T is a T_β-contractive mapping, then there exists a function $\beta\colon X \times X \to \mathbb{R}$ such that

$$\inf_{x \neq y \in X} \Big\{ d(x, y) - H(Tx, Ty) + \beta(x, y) \Big\} > 0.$$

We put

$$\gamma = \inf_{x \neq y \in X} \Big\{ d(x, y) - H(Tx, Ty) + \beta(x, y) \Big\},$$

hence for all $x \neq y \in X$, we get

$$H(Tx, Ty) - \beta(x, y) \leq d(x, y) - \gamma.$$

So

$$\alpha(x, y) e^{H(Tx, Ty)} \leq k e^{d(x, y)},$$

where $k = e^{-\gamma} < 1$ and $\alpha(x, y) = e^{-\beta(x, y)}$. Hence

$$\alpha(x, y) e^{H(Tx, Ty)} - k \leq k(e^{d(x, y)} - 1),$$

which implies that

$$\alpha(x, y) e^{H(Tx, Ty)} - e^{-\gamma} \leq k(e^{d(x, y)} - 1).$$

On the other hand, it follows from (iii) that $-\alpha(x, y) = -e^{-\beta(x, y)} \leq -e^{-\gamma}$. Then

$$\alpha(x, y) H_D(Tx, Ty) \leq k D(x, y),$$

for all $x, y \in X$, with $D(x, y) = e^{d(x, y)} - 1$ and H_D defined in Lemma 1.2.

Finally, we deduce from Lemma 2.1 and Lemma 2.2 that T has a fixed point $u \in X$. □

E x a m p l e 2.1. Let $X = \{0,1\}^2$ be endowed with the metric $d\big((x_1,y_1),(x_2,y_2)\big) = \|(x_1,y_1)-(x_2,y_2)\|_1 = |x_1-x_2|+|y_1-y_2|$. We note that (X,d) is not a uniform convex space, indeed:

For $\varepsilon = 1$, $x=(1,0)$ and $y=(0,1)$:

$\|x\|_1 = \|y\|_1 = 1$, $\|x-y\|_1 = 2 > 1 = \varepsilon$ and $\frac{1}{2}\|x+y\|_1 = 1 > 1-\delta$ for each $\delta > 0$.

Define the following multivalued mapping T by

$$T(0,0) = T(1,0) = T(0,1) = \{(0,0)\}, \quad T(1,1) = \{(1,0),(0,1),(0,0)\},$$

and a function $\beta\colon X\times X \to \mathbb{R}$ by

$$\begin{gathered}\beta((0,0),(0,0)) = \beta((0,0),(1,0)) = \beta((1,0),(0,0)) = 0,\\ \beta((0,0),(0,1)) = \beta((0,1),(0,0)) = 0,\\ \beta((1,0),(1,0)) = \beta((0,1),(0,1)) = 0,\\ \beta((1,1),(0,0)) = \beta((0,0),(1,1)) = \beta((0,1),(1,1)) = 1/3,\\ \beta((1,1),(1,0)) = \beta((1,0),(1,1)) = \beta((1,1),(1,0)) = 1/3,\end{gathered}$$

and

$$\beta((1,1),(1,1)) = 1/4.$$

So, we have, for all $x,y\in X$,

$$\inf_{x\neq y\in X}\Big\{d(x,y) - H(Tx,Ty) + \beta(x,y)\Big\} \geq 1/3.$$

Then T satisfies all conditions of Theorem 2.1 and has the fixed point $(0,0)$.

R e m a r k 2.1. Theorem 2.1 is a real and an important extension of Theorem 1.4. Indeed, the multivalued mapping defined in the above example is nonexpansive (i. e., $H(Tx,Ty)\leq d(x,y)$), therefore our result ensures the existence of a fixed point for a class of nonexpansive multivalued contractions without adding the uniform convexity of the space.

E x a m p l e 2.2. Let $X = \overline{B}(0,1)\times[0,1]$, where $\overline{B}(0,1)$ is the unit closed ball of a real Banach space of infinite dimension. Define the following metric on X by

$$d\big((x,y),(x',y')\big) = \begin{cases}1+|y-y'|, & \text{if } x\neq x',\\ |y-y'|, & \text{if } x = x'.\end{cases}$$

Define a multivalued mapping T by $T(x,y) = \overline{B}(0,1)\times\{1-y\}$ for all $(x,y)\in X$. Define a function $\beta\colon X\times X\to\mathbb{R}$ by

$$\beta((x,y),(x',y')) = \begin{cases}1/2, & \text{if } (x,y)\neq(x',y'),\\ 0, & \text{if } (x,y) = (x',y').\end{cases}$$

So, we have

$$\beta((x,y),(x',y'))\leq 0 \implies \beta^*(T(x,y),T(x',y'))\leq 0,$$

for all $(x,y),(x',y')\in X$. Then T satisfies all assumptions of Theorem 2.1 and T has the fixed point which is equal to $(0,1/2)$. Moreover, we observe that X is not compact and

$$\inf_{(x,y)\neq(x',y')\in X}\Big\{d\big((x,y),(x',y')\big) - H\big(T(x,y),T(x',y')\big) + \beta\big((x,y),(x',y')\big)\Big\} \geq 1/2 > 0.$$

Then T is a multivalued T_β-contractive mapping such that

$$\beta(a,b)\leq \inf_{(x,y)\neq(x',y')\in X}\Big\{d\big((x,y),(x',y')\big) - H\big(T(x,y),T(x',y')\big) + \beta\big((x,y),(x',y')\big)\Big\}$$

for all $a,b\in X$.

R e m a r k 2.2. Since for all $x, y \in X$ we have $d(x,y) - H(Tx,Ty) = 0$, Theorem 1.4 does not ensure the existence of the fixed point. Also, we note that X is not compact.

In the following, $\beta\colon X \times X \to \mathbb{R}$ is the function defined (on a metric space X) in the Definition 2.1 and Theorem 2.1. In order to state the second result, we first give the following definition.

D e f i n i t i o n 2.2. Φ is the class of all functions $\phi\colon [1,+\infty) \to \mathbb{R}$ satisfying:

(i) $\inf\limits_{t>1} \phi(t) > 0$;

(ii) $\phi(1) \leq \beta(x,x)$ for all $x \in X$.

D e f i n i t i o n 2.3. Let $T\colon X \to C(X)$ be a multivalued mapping of a metric space (X,d). T will be called a *T_β-weakly contractive multivalued mapping* if there exists $\phi \in \Phi$ such that

$$H(Tx,Ty) - \beta(x,y) \leq d(x,y) - \phi(1 + d(x,y)),$$

for all $x, y \in X$.

T h e o r e m 2.2. *Let $T\colon X \to C(X)$ be a T_β-weakly contractive multivalued mapping of a bounded complete metric space (X,d). Then T has a fixed point.*

P r o o f. Let $x \neq y \in X$, which implies by Definition 2.3 that

$$0 < \inf_{t>1} \phi(t) \leq \phi(1 + d(x,y)) \leq d(x,y) - H(Tx,Ty) + \beta(x,y).$$

Then

$$\inf_{x \neq y} \bigl\{ d(x,y) - H(Tx,Ty) + \beta(x,y) \bigr\} > 0.$$

By Theorem 2.1, T has a fixed point in X. □

E x a m p l e 2.3. Let $X = \{0,1,2\}$ with the usual metric $d(x,y) = |x-y|$ for all $x, y \in X$. Define a multivalued mapping T by

$$T0 = \{0\} \text{ and } T1 = \{0,1\} = T2,$$

a function $\beta : X \times X \to \mathbb{R}$ by

$$\beta(x,y) = \begin{cases} 1/2, & \text{if } x = y, \\ 3/2, & \text{if } x \neq y, \end{cases}$$

and a function $\phi\colon [1,\infty) \to \mathbb{R}$ by

$$\phi(t) = \begin{cases} -1, & \text{if } t = 1, \\ 1, & \text{if } t > 1. \end{cases}$$

We have $\phi(1) = -1 \leq \beta(x,x)$ for all $x \in X$. Further,

$$\beta(a,b) \leq 3/2 \leq \inf_{x \neq y \in X} \left\{ d(x,y) - H(Tx,Ty) + \beta(x,y) \right\}$$

for all $a, b \in X$.

Hence, we have the following cases:

Case1: $H(T0,T1) - \beta(0,1) = -1/2 \leq 0 = d(0,1) - \phi(1 + d(0,1))$.

Case2: $H(T0,T2) - \beta(0,2) = -1/2 \leq 1 = d(0,2) - \phi(1 + d(0,2))$.

Case3: $H(T1,T2) - \beta(1,2) = -3/2 \leq 0 = d(1,2) - \phi(1 + d(1,2))$.

Then, T satisfies all assumptions in Theorem 2.2 and $0 \in T0$.

§ 3. Applications

§ 3.1. Integral inclusion

Different fixed point theorems have been used in several contexts to ensure the existence of a solution for integral inclusions (see [10, 19, 23, 24, 27, 29]). Throughout this section we assume that $X = \mathcal{C}([0,\tau],\mathbb{R})$ is the space of all continuous functions from $[0,\tau]$ ($\tau > 0$) into $\mathbb{R}$ and $\mathcal{P}_{cv}(\mathbb{R})$ is the class of nonempty compact and convex subsets of $\mathbb{R}$. If X is equipped with $d(x,y) = \sup\limits_{t\in[0,\tau]} |x(t) - y(t)|$, then (X,d) is a complete metric space. In this section, we first consider the following Volterra-type inclusion

$$x(t) \in f(t) + \int_0^t K(t,s,x(s))\,ds, \quad t \in [0,\tau], \tag{3.1}$$

where $f \in X$ and $K\colon [0,\tau]\times[0,\tau]\times\mathbb{R} \to \mathcal{P}_{cv}(\mathbb{R})$. We suppose that the multi-valued mapping $K_x(t,s) := K(t,s,x(s))$, $(t,s) \in [0,\tau]^2$ is lower semicontinuous for each $x \in X$.

Now, our main purpose is to weaken some conditions of Theorem 5 [29], so we define a multivalued operator T by

$$Tx(t) = \left\{ v \in X\colon v(t) \in f(t) + \int_0^t K(t,s,x(s))\,ds,\ t\in[0,\tau] \right\},$$

for all $x \in X$.

According to Michael's selection Theorem [19], for all $x \in X$ there exists a continuous operator $k_x\colon [0,\tau]\times[0,\tau] \to \mathbb{R}$ (selection) such that $k_x(t,s) \in K_x(t,s)$ for any $t,s \in [0,\tau]$ and hence $f(t) + \int_0^t k_x(t,s)\,ds \in Tx(t)$ which leads to $Tx \neq \emptyset$. Moreover, Tx is a closed set (see [27, 29]).

We now have all the tools needed to prove the next result.

T h e o r e m 3.1. *Suppose that there exist $M > 0$ and a function $\eta\colon X\times X \to \mathbb{R}$ such that for all $s,t \in [0,\tau]$, $n \in \mathbb{N}$, and $x \neq y \in X$ the following hypotheses hold:*

(i)

$$\begin{aligned} \eta(x,y) \ge 0 &\implies H(K(t,s,x(s)),K(t,s,y(s))) \le \frac{1}{\tau}[|x(s)-y(s)| - M],\\ \eta(x,y) < 0 &\implies H(K(t,s,x(s)),K(t,s,y(s))) \le \frac{1}{\tau}|x(s)-y(s)|; \end{aligned} \tag{3.2}$$

(ii) $\eta(x,y) \ge 0 \implies \eta_*(Tx,Ty) \ge 0$, *with* $\eta_*(Tx,Ty) = \inf\{\eta(a,b)\colon a \in Tx,\ b \in Ty\}$;

(iii) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\eta(x_0,x_1) \ge 0$;*

(iv) *if $\{x_n\} \subset X$ is a sequence convergent to $x \in X$ with $\eta(x_n,x_{n+1}) \ge 0$, then $\eta(x_n,x) \ge 0$.*

Then the integral inclusion (3.1) *has a solution.*

P r o o f. Let $x,y \in X$ be such that $a \in Tx$. Hence, there exists $k_x(t,s) \in K_x(t,s)$ for $t,s \in [0,\tau]$ with $a(t) = f(t) + \int_0^t k_x(t,s)\,ds$. So, the condition (3.2) implies that there exists $b(t,s) \in K_y(t,s)$ such that

$$\begin{aligned} \eta(x,y) \ge 0 &\implies |k_x(t,s) - b(t,s)| \le \frac{1}{\tau}[|x(s)-y(s)| - M],\\ \eta(x,y) < 0 &\implies |k_x(t,s) - b(t,s)| \le \frac{1}{\tau}|x(s)-y(s)|, \end{aligned}$$

for all $t,s \in [0,\tau]$.

In this step, let us consider the set-valued map S defined as follows:

if $\eta(x,y) \geq 0 \implies$
$$S(t,s) = K_y(t,s) \cap \{w \in \mathbb{R} \colon |k_x(t,s) - w| \leq \frac{1}{\tau}[|x(t,s) - y(t,s)| - M]\},$$
if $\eta(x,y) < 0 \implies$
$$S(t,s) = K_y(t,s) \cap \{w \in \mathbb{R} \colon |k_x(t,s) - w| \leq \frac{1}{\tau}|x(t,s) - y(t,s)|\},$$
for all $t, s \in [0,\tau]$.

Since S is a lower semicontinuous multivalued mapping, then, in view of Michael's selection theorem, there exists a continuous mapping $k_y \colon [0,\tau] \times [0,\tau] \to \mathbb{R}$ such that $k_y(t,s) \in S(t,s)$, for all $t, s \in [0,\tau]$. Therefore, we get
$$c(t) = f(t) + \int_0^t k_y(t,s)\,ds \in f(t) + \int_0^t K(t,s,y(s))\,ds, \quad t \in [0,\tau],$$
and for any $t \in [0,\tau]$, we have
$$|a(t) - c(t)| = \left| \int_0^t k_x(t,s)\,ds - \int_0^t k_y(t,s)\,ds \right|.$$
Then, we have:
$$\text{if } \eta(x,y) \geq 0 \implies |a(t) - c(t)| \leq \int_0^t |k_x(t,s) - k_y(t,s)|\,ds \leq d(x,y) - M,$$
$$\text{if } \eta(x,y) < 0 \implies |a(t) - c(t)| \leq \int_0^t |k_x(t,s) - k_y(t,s)|\,ds \leq d(x,y),$$
which implies that:
$$\text{if } \eta(x,y) \geq 0 \implies d(a,c) \leq d(x,y) - M,$$
$$\text{if } \eta(x,y) < 0 \implies d(a,c) \leq d(x,y),$$
for all $x \neq y \in X$.

Now, define $\beta \colon X \times X \to \mathbb{R}$ by
$$\beta(x,y) = \begin{cases} 0, & \text{if } \eta(x,y) \geq 0, \\ 1, & \text{otherwise.} \end{cases}$$
It is straightforward to check the implication
$$\beta(x,y) \leq 0 \implies \beta_*(x,y) \leq 0.$$
Also, by interchanging the role of x and y and by using Lemma 1.1, we conclude that
$$\inf_{x \neq y \in X} \{d(x,y) - H(Tx,Ty) + \beta(x,y)\} > 0,$$
and hence T is a T_β-contractive multivalued mapping. Moreover, it is easy to see that the conditions (i), (ii) and (iii) of Theorem 2.1 are satisfied. This theorem ensures the existence of a solution for the Volterra-type integral inclusion (3.1). □

Remark 3.1. If we take $\eta(x,y) = 0$ in Theorem 3.1 we obtain the statement of Theorem 5 proved in [29] (that is: if there exists $M > 0$ such that for all $s, t \in [0,\tau]$ and $x \neq y \in X$ we have $H(K(t,s,x(s)), K(t,s,y(s))) \leq \frac{1}{\tau}[|x(s) - y(s)| - M]$, then the integral inclusion (3.1) has a solution). Therefore, our result is a real enhancement of this theorem.

§ 3.2. Approximation of the fixed point via a new and faster iterative process

In 2000, Noor [21] introduced the following iterative process for a mapping $T\colon X \to X$ by

$$\begin{cases} t_1 = t \in X, \\ t_{n+1} = (1-\alpha_n)t_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)t_n + \beta_n T z_n, \\ z_n = (1-\gamma_n)t_n + \gamma_n T t_n, \qquad n \in \mathbb{N}, \end{cases} \tag{3.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$.

In 2019, Okeke [22] gave the so-called Picard–Krasnoselskii iterative process as follows

$$\begin{cases} x_1 = x \in X, \\ x_{n+1} = T v_n, \\ v_n = (1-\alpha_n)x_n + \alpha_n T u_n, \\ u_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \in \mathbb{N}, \end{cases} \tag{3.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $(0,1)$. The author in [22] showed that this iterative scheme converges faster than the scheme introduced by Noor [21] and other known schemes in the literature [15, 17, 18].

In the following, we give some needed definitions for our approach.

D e f i n i t i o n 3.1 (see [5]). Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers converging to a and b respectively. If

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = 0,$$

then $\{a_n\}$ converges faster than $\{b_n\}$.

D e f i n i t i o n 3.2 (see [5]). Suppose that for two fixed point iterative processes $\{u_n\}$ and $\{v_n\}$, both converging to the same fixed point p, the error estimates

$$\|u_n - p\| \le a_n, \quad \|v_n - p\| \le b_n \quad \text{for all } n \in \mathbb{N},$$

are satisfied where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers converging to zero. If $\{a_n\}$ converges faster than $\{b_n\}$, then $\{u_n\}$ converges faster than $\{v_n\}$ to p.

Throughout this section, let $(X, \|\cdot\|)$ be a bounded Banach space. For $x \in X$, we denote by $P_T(x) = \{y \in Tx\colon \|x-y\| = d(x, Tx)\}$ a multivalued mapping $T\colon X \to C(X)$. Now, we introduce the multivalued version of the scheme (3.4) as follows

$$\begin{cases} x_1 = x \in X, \\ z_n = (1-\beta_n)x_n + \beta_n u_n, \\ y_n = (1-\alpha_n)x_n + \alpha_n v_n, \\ x_{n+1} = w_n, \qquad n \in \mathbb{N}, \end{cases} \tag{3.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $(0,1)$ such that

$$u_n \in P_T(x_n), \quad v_n \in P_T(z_n) \text{ and } w_n \in P_T(y_n).$$

Also, we can give the multivalued version of the Noor iterative process (3.3) as follows:

$$\begin{cases} t_1 = t \in X, \\ t_{n+1} = (1-\alpha_n)t_n + \alpha_n a_n, \\ y_n = (1-\beta_n)t_n + \beta_n b_n, \\ z_n = (1-\gamma_n)t_n + \gamma_n c_n, \qquad n \in \mathbb{N}, \end{cases} \tag{3.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$ such that

$$a_n \in P_T(y_n), \quad b_n \in P_T(z_n) \ \text{ and } \ c_n \in P_T(t_n).$$

Now, we are able to state the following result

T h e o r e m 3.2. *Let X be a real Banach space such that $T\colon X \to C(X)$ with $F(T) :=$ $:= \{x \in X : x \in Tx\} \neq \emptyset$. We suppose that P_T is a T-weakly contractive multivalued mapping* (*see Theorem* 1.5). *Let $\{x_n\}$ be a sequence generated by the iteration scheme* (3.5) *such that $\sum_n \alpha_n\beta_n = \infty$. Then $\{x_n\}$ converges to $u \in F(T)$.*

P r o o f. Since $F(T) \neq \emptyset$, let $u \in F(T)$. From (3.5) it follows that

$$\|x_{n+1} - u\| = \|w_n - u\| \le H(P_T(y_n), P_T(u)) \le \|y_n - u\| - \phi(1 + \|y_n - u\|) \le \|y_n - u\|. \tag{3.7}$$

From the assumptions that P_T is a T-weakly contractive multivalued mapping and (3.5), we obtain

$$\begin{aligned} \|y_n - u\| &= \|(1-\alpha_n)x_n + \alpha_n v_n - u\| \\ &\le (1-\alpha_n)\|x_n - u\| + \alpha_n\|v_n - u\| \\ &\le (1-\alpha_n)\|x_n - u\| + \alpha_n H(P_T(z_n), P_T(u)) \\ &\le (1-\alpha_n)\|x_n - u\| + \alpha_n\|z_n - u\| - \alpha_n\phi(1 + \|z_n - u\|) \\ &\le (1-\alpha_n)\|x_n - u\| + \alpha_n\|z_n - u\|. \end{aligned} \tag{3.8}$$

We have also

$$\begin{aligned} \|z_n - u\| &= \|(1-\beta_n)x_n + \beta_n u_n - u\| \\ &\le (1-\beta_n)\|x_n - u\| + \beta_n\|u_n - u\| \\ &\le (1-\beta_n)\|x_n - u\| + \beta_n H(P_T(x_n), P_T(u)) \\ &\le (1-\beta_n)\|x_n - u\| + \beta_n\|x_n - u\| - \beta_n\phi(1 + \|x_n - u\|) \\ &\le \|x_n - u\|. \end{aligned} \tag{3.9}$$

By combining with (3.8) and (3.9), we get

$$\|y_n - u\| \le (1-\alpha_n)\|x_n - u\| + \alpha_n\|x_n - u\| \le \|x_n - u\|. \tag{3.10}$$

From (3.7) and (3.10), we have

$$\|x_{n+1} - u\| \le \|x_n - u\|. \tag{3.11}$$

Therefore, $\{d_n = \|x_n - u\| + 1\}$ converges to $d \ge 1$. Now, from (3.7), (3.10) and (3.11), we get

$$d_{n+1} \le d_n - \phi(d'_n), \tag{3.12}$$

where $d'_n = 1 + \|y_n - u\|$.

In the same manner, we can show that $\{d'_n\}$ converges to $d' \ge 1$.

If $d' = 1$, so, by (3.7), $\{x_n\}$ converges to u.

If $d' > 1$, the inequality (3.12) implies that $\phi(d'_n) \le d_n - d_{n+1}$. Since $0 < \alpha_n\beta_n < 1$, we have

$$\sum_{n=1}^{\infty} \alpha_n\beta_n \inf_{n\in\mathbb{N}} \phi(d'_n) \le \sum_{n=1}^{\infty} \alpha_n\beta_n\phi(d'_n) \le \sum_{n=1}^{\infty} (d_n - d_{n+1}) \le d_1 - d.$$

This leads to a contradiction with the fact that $\sum\limits_{n=1}^{\infty} \alpha_n\beta_n = \infty$. Therefore $d' = 1$, and hence $d = 1$ which implies that $\{x_n\}$ converges to u. □

In the following, we take $\phi(1+t) = 1+t$ and we prove that our new scheme converges faster than the multivalued version of the Noor iterative process in the sense of Definition 3.2.

T h e o r e m 3.3. *The new introduced scheme* (3.5) *converges faster than the multivalued version of the Noor iterative process* (3.6).

P r o o f. From (3.5) and the proof of Theorem 3.2, we obtain

$$\|x_{n+1}-u\| \leq \|x_n-u\|-\alpha_n\beta_n\phi(1+\|x_n-u\|) \leq (1-\alpha_n\beta_n)\|x_n-u\| \leq \ldots \leq (1-\alpha_n\beta_n)^n\|x_1-u\|,$$

for all $n \in \mathbb{N}$. Let

$$A_n = (1-\alpha_n\beta_n)^n\|x_1-u\|. \tag{3.13}$$

On the other hand, from (3.6) it follows that

$$\begin{aligned} \|t_{n+1}-u\| &\leq (1-\alpha_n)\|t_n-u\| + \alpha_n\|a_n-u\| \\ &\leq (1-\alpha_n)\|t_n-u\| + \alpha_n H(P_T(y_n), P_T(u)) \\ &\leq (1-\alpha_n)\|t_n-u\| + \alpha_n\|y_n-u\|. \end{aligned} \tag{3.14}$$

Also by (3.6), we have

$$\begin{aligned} \|y_n-u\| &\leq (1-\beta_n)\|t_n-u\| + \beta_n\|b_n-u\| \\ &\leq (1-\beta_n)\|t_n-u\| + \beta_n H(P_T(z_n), P_T(u)) \\ &\leq (1-\beta_n)\|t_n-u\| + \beta_n\|z_n-u\| \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \|z_n-u\| &\leq \|(1-\gamma_n)\|t_n-u\| + \gamma_n\|c_n-u\| \\ &\leq (1-\gamma_n)\|t_n-u\| + \gamma_n H(P_T(t_n), P_T(u)) \\ &\leq \|t_n-u\| - \gamma_n\|t_n-u\|. \end{aligned} \tag{3.16}$$

By combining with (3.14), (3.15) and (3.16), we get

$$\|t_{n+1}-u\| \leq (1-\alpha_n\beta_n\gamma_n)\|t_n-u\| \leq \ldots \leq (1-\alpha_n\beta_n\gamma_n)^n\|t_1-u\|,$$

for all $n \in \mathbb{N}$. Let

$$B_n = (1-\alpha_n\beta_n\gamma_n)^n\|t_1-u\|. \tag{3.17}$$

Since $\alpha_n\beta_n\gamma_n < \alpha_n$, we conclude from (3.13) and (3.17) that $\lim \dfrac{A_n}{B_n} = 0$. □

R e m a r k 3.2. In this application, the iterative process is more general than Noor's iterative process (see [21]) because of the following two reasons:

(1) the convergence is obtained only in a Banach space; compared to Noor iterative process, the convergence is proved over a Hilbert space; in other words, we didn't employ the inner product in any step;
(2) we get the convergence in the setting of multivalued mappings, while Noor iterative process is achieved only for single-valued mappings.

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Received 21.04.2023

Accepted 23.02.2024

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Citation: Y. Touail. Set-valued mappings: fixed point results with β-function and some applications, *Izvestiya Instituta Matematiki i Informatiki Udmurtskogo Gosudarstvennogo Universiteta*, 2024, vol. 63, pp. 61–79.

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Многозначные отображения: результаты о неподвижной точке с использованием β-функции и некоторые приложения

Ключевые слова: неподвижная точка, многозначные отображения, T_β-многозначное сжимающее отображение, T_β-слабо сжимающее многозначное отображение, интегральное включение, итерационный процесс.

УДК: 517.98

DOI: 10.35634/2226-3594-2024-63-05

С помощью так называемой β-функции получены некоторые результаты о неподвижной точке для нерасширяющих многозначных отображений. В данной работе результаты рассматриваются в контексте полных метрических пространств, которые не являются ни равномерно выпуклыми, ни компактными. Полученные результаты расширяют, объединяют и улучшают несколько недавних результатов в существующей литературе. В заключение мы применяем наши новые результаты, чтобы обеспечить существование решения для нелинейного интегрального включения. Кроме того, мы аппроксимируем неподвижную точку более быстрым итерационным процессом.

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Поступила в редакцию 21.04.2023

Принята к публикации 23.02.2024

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Цитирование: Ю. Туаиль. Многозначные отображения: результаты о неподвижной точке с использованием β-функции и некоторые приложения // Известия Института математики и информатики Удмуртского государственного университета. 2024. Т. 63. С. 61–79.